

Grassmann variables on quantum spaces

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Abstract. Attention is focused on antisymmetrized versions of quantum spaces that are of particular importance in physics, i.e. two-dimensional quantum plane, q -deformed Euclidean space in three or four dimensions as well as q -deformed Minkowski space. For each case standard techniques for dealing with q -deformed Grassmann variables are developed. Formulae for multiplying supernumbers are given. The actions of symmetry generators and fermionic derivatives upon antisymmetrized quantum spaces are calculated. The complete Hopf structure for all types of quantum space generators is written down. From the formulae for the coproduct a realization of the L -matrices in terms of symmetry generators can be read off. The L -matrices together with the action of symmetry generators determine how quantum spaces of different type have to be fused together.

1 Introduction

It is an old idea that limiting the precision of position measurements by a fundamental length will lead to a new method for regularizing quantum field theories [1]. It is also well-known that such a modification of classical spacetime will in general break its Poincaré symmetry [2]. One way out of this difficulty is to change not only spacetime, but also its underlying symmetry.

Quantum groups can be seen as deformations of classical spacetime symmetries, as they describe the symmetry of their comodules, the so-called quantum spaces. From a physical point of view the most realistic examples for quantum groups and quantum spaces arise from q -deformation [3–9]. In our work we are interested in q -deformed versions of Minkowski space and Euclidean spaces as well as their corresponding symmetries, given by q -deformed Lorentz algebra and algebras of q -deformed angular momentum, respectively [10–14]. Wess and his coworkers were able to show that q -deformation of spaces and symmetries can indeed lead to discretizations, as they result from the existence of a smallest distance [15, 16]. This observation nourishes the hope that q -deformation might give a new method to regularize quantum field theories [17–20].

In our previous work [21–26] attention was focused on symmetrized versions of q -deformed quantum spaces that are of particular importance in physics, i.e. two-dimensional Manin plane, q -deformed Euclidean space in three or four dimensions, and q -deformed Minkowski space. As there is a need for Grassmann variables in physics we would like to

discuss antisymmetrized versions of those quantum spaces as well.

In particular, we intend to proceed as follows. In Sect. 2 we cover the ideas our considerations about q -deformed quantum spaces are based on. For further details we recommend [27, 28]. In the subsequent sections we apply these reasonings to antisymmetrized versions of two-dimensional quantum plane, q -deformed Euclidean space with three or four dimensions as well as q -deformed Minkowski-space.

More concretely, we develop some standard techniques for dealing with q -deformed Grassmann variables. In doing so, we start from the commutation relations for q -deformed Grassmann variables and introduce q -deformed supernumbers. After that we are going to derive explicit formulae for multiplying q -deformed supernumbers. In addition to this, we are going to calculate the action of symmetry generators and partial derivatives upon antisymmetrized quantum spaces. Furthermore, we are going to write down the complete Hopf structure on quantum space generators, including their coproduct, antipode, and counit.

One should realize that the explicit form of the coproduct on quantum space generators enables us to read off a realization of the so-called L -matrices in terms of symmetry generators. This knowledge together with the action of symmetry generators upon quantum spaces tells us how quantum spaces of different type have to be fused together.

2 Basic ideas on antisymmetrized quantum spaces

In our approach spacetime symmetries are described by quantum algebras like $U_q(su_2)$, $U_q(so_4)$ or q -deformed Lorentz algebra. Important for us is the fact that these alge-

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bras are *quasitriangular Hopf algebras*, i.e. their coproduct can be twisted by an invertible element $\mathcal{R} \in H \otimes H$, which is known as the universal R -matrix of the corresponding Hopf algebra H . Formally, we have

$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}, \quad h \in H, \quad (1)$$

where Δ and τ denote respectively the coproduct on H and the transposition map.

The modules of the quantum algebras are called *quantum spaces*. At a first glance a quantum space is nothing other than an algebra A generated by non-commuting coordinates X_1, X_2, \dots, X_n , i.e.

$$A = \mathbb{C}[[X_1, \dots, X_n]]/\mathcal{I}, \quad (2)$$

where \mathcal{I} denotes the ideal generated by the relations of the non-commuting coordinates.

It should be noted that we can combine a quantum algebra H with its representation space A to form a *left cross product algebra* $A \rtimes H$ built on $A \otimes H$ with product

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b) \otimes h_{(2)}g, \quad a, b \in A, \quad h, g \in H, \quad (3)$$

where \triangleright denotes the left action of H on A . There is also a right-handed version of this notion called a *right cross product algebra* $H \ltimes A$ and built on $H \otimes A$ with product

$$(h \otimes a)(g \otimes b) = hg_{(2)} \otimes (a \triangleleft g_{(1)})b, \quad (4)$$

where \triangleleft now stands for the right action of H on A . The last two identities tell us that the commutation relations between symmetry generators and representation space elements are completely determined by coproduct and action of the symmetry generators, since we obtain from them

$$hb = (h_{(1)} \triangleright b)h_{(2)}, \quad ag = g_{(2)}(a \triangleleft g_{(1)}). \quad (5)$$

However, in what follows it is necessary to take another point of view which is provided by category theory. A category is a collection of objects X, Y, Z, \dots together with a set $\text{Mor}(X, Y)$ of morphisms between two objects X, Y . The composition of morphisms has similar properties as the composition of maps. We are interested in tensor categories. These categories have a product, denoted \otimes and called the tensor product. It admits several ‘‘natural’’ properties such as associativity and existence of a unit object. For a more formal treatment we refer to [27, 29–31]. If the action of a quasitriangular Hopf algebra H on the tensor product of two quantum spaces X and Y is defined by

$$h \triangleright (v \otimes w) = (h_{(1)} \triangleright v) \otimes (h_{(2)} \triangleright w) \in X \otimes Y, \quad h \in H, \quad (6)$$

where the coproduct is written in the so-called Sweedler notation, i.e. $\Delta(h) = h_{(1)} \otimes h_{(2)}$, then the representations (quantum spaces) of the given Hopf algebra (quantum algebra) are the objects of a tensor category. In this tensor category exist a number of morphisms of particular importance that are covariant with respect to the Hopf algebra action. First of all, for any pair of objects X, Y there is an isomorphism $\Psi_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ such that

$(g \otimes f) \circ \Psi_{X,Y} = \Psi_{X',Y'} \circ (f \otimes g)$ for arbitrary morphisms $f \in \text{Mor}(X, X')$ and $g \in \text{Mor}(Y, Y')$. In addition to this one requires the hexagon axiom to hold. The hexagon axiom is the validity of the two conditions

$$\Psi_{X,Z} \circ \Psi_{Y,Z} = \Psi_{X \otimes Y, Z}, \quad \Psi_{X,Z} \circ \Psi_{X,Y} = \Psi_{X, Y \otimes Z}. \quad (7)$$

A tensor category equipped with such mappings $\Psi_{X,Y}$ for each pair of objects X, Y is called a *braided tensor category*. The mappings $\Psi_{X,Y}$ as a whole are often referred to as the braiding of the tensor category. Furthermore, for any quantum space algebra X in this category there are morphisms $\Delta : X \rightarrow X \otimes X$, $S : X \rightarrow X$, and $\varepsilon : X \rightarrow \mathbb{C}$ forming a braided Hopf algebra, i.e. Δ , S , and ε obey the usual axioms of a Hopf algebra, but now as morphisms in the braided category.

It is well-known that for a quasitriangular Hopf algebra H the category of H -modules is braided, with

$$\Psi_{X,Y}(v \otimes w) = (\mathcal{R}^{(2)} \triangleright w) \otimes (\mathcal{R}^{(1)} \triangleright v), \quad v \in X, w \in Y, \quad (8)$$

where $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$. In terms of quantum space generators the above identity becomes

$$\Psi_{X,Y}(X^i \otimes Y^j) = \hat{R}_{kl}^{ij} Y^k \otimes X^l, \quad (9)$$

where summation over repeated indices is to be understood. Notice that the matrix \hat{R} describes nothing other than a linear mapping between vector spaces spanned by tensor products of quantum space generators. In the cases under consideration this mapping can be restricted to invariant subspaces. As a consequence, \hat{R} admits a projector decomposition of the general form [32, 33]

$$\hat{R} = \alpha_S P_S + \alpha_A P_A + \alpha_T P_T, \quad (10)$$

where α_S , α_A , and α_T denote the corresponding eigenvalues. The projectors P_S and P_A are quantum analogs of a symmetrizer and an antisymmetrizer, respectively, while P_T projects onto a one-dimensional subspace generated by the quantum length.

The relations of the *quantum symmetric space* are determined by [34]

$$(P_A)_{kl}^{ij} X^k X^l = 0, \quad (11)$$

and likewise for the *quantum antisymmetric space* (*q-deformed Grassmann algebra*),

$$(P_S)_{kl}^{ij} \theta^k \theta^l = 0, \quad (P_T)_{kl}^{ij} \theta^k \theta^l = 0. \quad (12)$$

Alternatively, the two identities defining quantum antisymmetric space can be combined in the following way:

$$\begin{aligned} \theta^i \theta^j &= \left((P_S)_{kl}^{ij} + (P_A)_{kl}^{ij} + (P_T)_{kl}^{ij} \right) \theta^k \theta^l \\ &= (P_A)_{kl}^{ij} \theta^k \theta^l = \alpha_A^{-1} \hat{R}_{kl}^{ij} \theta^k \theta^l. \end{aligned} \quad (13)$$

Let us mention that the quantum spaces obtained this way satisfy the so-called *Poincaré–Birkhoff–Witt property*,

i.e. the dimension of a subspace of homogenous polynomials should be the same as for the corresponding classical variables. This property is the deeper reason why normal ordered monomials again constitute a basis of q -deformed Grassmann algebras. Consequently, each q -deformed supernumber can be represented in the general form

$$f(\underline{\theta}) = f' + \sum f_K \theta^K, \tag{14}$$

where the f are arbitrary complex numbers and the θ^K stand for monomials of a given normal ordering.

Next, we want to deal with the covariant differential calculus on quantum spaces [35–37]. Such a differential calculus can be established by introducing an exterior derivative d with the usual properties of nilpotency and Leibniz rule:

$$d^2 = 0, \tag{15}$$

$$d(fg) = (df)g + (-1)^{|f|} f(dg),$$

where

$$|f| = \begin{cases} 0, & \text{if } f \text{ bosonical,} \\ 1, & \text{if } f \text{ fermionical.} \end{cases} \tag{16}$$

In addition to this, we require that the differentials of the coordinates,

$$\xi^i \equiv dX^i, \quad \eta^i \equiv d\theta^i, \tag{17}$$

are subject to the relations

$$(P_S)^{ij}_{kl} \xi^k \xi^l = 0, \quad (P_T)^{ij}_{kl} \xi^k \xi^l = 0, \tag{18}$$

$$(P_A)^{ij}_{kl} \eta^k \eta^l = 0, \quad (P_T)^{ij}_{kl} \eta^k \eta^l = 0. \tag{19}$$

With the same reasonings already applied in (13) the above identities lead to

$$\xi^i \xi^j = \alpha_A^{-1} \hat{R}_{kl}^{ij} \xi^k \xi^l, \quad \eta^i \eta^j = \alpha_S^{-1} \hat{R}_{kl}^{ij} \eta^k \eta^l. \tag{20}$$

In order to find commutation relations between coordinates and differentials, we make as ansatz

$$X^i \xi^j = B_{kl}^{ij} \xi^k X^l, \quad \theta^i \eta^j = C_{kl}^{ij} \eta^k \theta^l. \tag{21}$$

Applying the exterior derivative to both sides of the above equations and comparing the results with (20) then yields for the unknown coefficients

$$B_{kl}^{ij} = -\alpha_A^{-1} \hat{R}_{kl}^{ij}, \quad C_{kl}^{ij} = \alpha_S^{-1} \hat{R}_{kl}^{ij}. \tag{22}$$

As a next step we introduce partial derivatives by

$$d = \xi^i (\partial_x)_i \quad \text{and} \quad d = \eta^i (\partial_\theta)_i. \tag{23}$$

From (15) together with (21) it can be shown that the following Leibniz rules hold:

$$(\partial_x)_i X^j = \delta_i^j - \alpha_A^{-1} \hat{R}_{il}^{jk} X^l (\partial_x)_k, \tag{24}$$

$$(\partial_\theta)_i \theta^j = \delta_i^j - \alpha_S^{-1} \hat{R}_{il}^{jk} \theta^l (\partial_\theta)_k.$$

We could also have started our considerations from the inverse braiding

$$\Psi_{X,Y}^{-1}(v \otimes w) = \left((\mathcal{R}^{-1})^{(1)} \triangleright w \right) \otimes \left((\mathcal{R}^{-1})^{(2)} \triangleright v \right), \tag{25}$$

leading us to

$$\Psi_{X,Y}^{-1}(X^i \otimes Y^j) = (\hat{R}^{-1})^{ij}_{kl} Y^k \otimes X^l. \tag{26}$$

\hat{R}^{-1} denotes the inverse of \hat{R} , so its projector decomposition is given by

$$\hat{R}^{-1} = \alpha_S^{-1} P_S + \alpha_A^{-1} P_A + \alpha_T^{-1} P_T. \tag{27}$$

Repeating the same steps as before we get relations for conjugated objects. However, their explicit form can be obtained from the above relations most easily by applying the substitutions

$$\hat{R} \rightarrow \hat{R}^{-1}, \quad \alpha_{A,S,T} \rightarrow \alpha_{A,S,T}^{-1}, \tag{28}$$

$$\begin{aligned} \partial_x^i &\rightarrow \hat{\partial}_x^i, & \partial_\theta^i &\rightarrow \hat{\partial}_\theta^i, \\ a^i &\rightarrow \bar{a}^i, & a^i &\in \{\xi^i, \eta^i, X^i, \theta^i\}. \end{aligned}$$

Lastly, let us say a few words about the Hopf structures on quantum spaces. With the L -matrix and its conjugate [38], which can be introduced by

$$\Psi_{X,Y}(a^i \otimes w) = ((\mathcal{L}_a)_j^i \triangleright w) \otimes a^j, \tag{29}$$

$$\Psi_{X,Y}^{-1}(a^i \otimes w) = ((\bar{\mathcal{L}}_a)_j^i \triangleright w) \otimes a^j,$$

the two Hopf structures on quantum space generators can be written as [39]

$$\Delta(a^i) = a^i \otimes 1 + (\mathcal{L}_a)_j^i \otimes a^j, \tag{30}$$

$$\bar{\Delta}(a^i) = a^i \otimes 1 + (\bar{\mathcal{L}}_a)_j^i \otimes a^j,$$

$$S(a^i) = -S(\mathcal{L}_a)_j^i a^j, \quad \bar{S}(a^i) = -S(\bar{\mathcal{L}}_a)_j^i a^j, \tag{31}$$

$$\varepsilon(a^i) = \bar{\varepsilon}(a^i) = 0. \tag{32}$$

One should notice that the entries of the L -matrices live in the corresponding quantum algebra H . This way, we can conclude that the above expressions are part of the Hopf structure of the crossed product algebra $A \rtimes H$.

3 Two-dimensional quantum plane

We begin by describing the two-dimensional antisymmetrized quantum plane algebra explicitly. For this purpose we need the projector decomposition of the R -matrix for $U_q(su_2)$ [32]:

$$\hat{R} = qP_S - q^{-1}P_A. \tag{33}$$

One should notice that in this case P_A and P_T coincide, so we have only two different projectors in (33). For the antisymmetrized coordinates, the decomposition in (33) implies [cf. (13)]

$$\theta^i \theta^j = -q \hat{R}_{kl}^{ij} \theta^k \theta^l. \tag{34}$$

Inserting the explicit form for the R -matrix [33], we get from (34) the following independent relations:

$$(\theta^1)^2 = (\theta^2)^2 = 0, \quad \theta^1\theta^2 = -q^{-1}\theta^2\theta^1. \quad (35)$$

To go further, we introduce supernumbers, which we can write in the form

$$f(\theta^1, \theta^2) = f' + f_1\theta^1 + f_2\theta^2 + f_{12}\theta^1\theta^2. \quad (36)$$

Using relations (35) it is not very difficult to show that the product of two supernumbers can be written as

$$\begin{aligned} (f \cdot g)(\theta^1, \theta^2) & \quad (37) \\ &= (f \cdot g)' + (f \cdot g)_1\theta^1 + (f \cdot g)_2\theta^2 + (f \cdot g)_{12}\theta^1\theta^2, \end{aligned}$$

with

$$\begin{aligned} (f \cdot g)' &= f'g', & (38) \\ (f \cdot g)_i &= f_i g' + f' g_i, \quad i = 1, 2, \\ (f \cdot g)_{12} &= f_1 g_2 - q f_2 g_1 + f' g_{12} + f_{12} g'. \end{aligned}$$

Next, we come to the action of symmetry generators on supernumbers. To this end, we have to recall that both bosonic and fermionic coordinates transform as spinors under the action of the symmetry algebra $U_q(su_2)$. Using for $U_q(su_2)$ the form as it was introduced in [14] the commutation relations between its independent generators (denoted by T^+ , T^- , and τ) and the spinor components a^i , $i = 1, 2$, read

$$T^+ a^1 = qa^1 T^+ + q^{-1} a^2, \quad (39)$$

$$T^+ a^2 = q^{-1} a^2 T^+, \quad T^- a^1 = qa^1 T^-, \quad (40)$$

$$T^- a^2 = q^{-1} a^2 T^- + qa^1, \quad \tau a^1 = q^2 a^1 \tau, \quad (41)$$

$$\tau a^2 = q^{-2} a^2 \tau.$$

From the above relations we can derive the action of the symmetry generators on a supernumber of the form (36). To this end, we repeatedly apply the commutation relations (39)–(41) to the product of a symmetry generator and a supernumber, until we obtain an expression with all symmetry generators standing to the right of all quantum plane coordinates. In doing so, we get the left action of a symmetry generator on a supernumber. Explicitly, we find

$$T^+ \triangleright f(\theta^1, \theta^2) = q^{-1} f_1 \theta^2, \quad (42)$$

$$T^- \triangleright f(\theta^1, \theta^2) = q f_2 \theta^1,$$

$$\tau \triangleright f(\theta^1, \theta^2) = f(q^2 \theta^1, q^{-2} \theta^2).$$

Right actions of symmetry generators on supernumbers can be derived in a similar way, if we now consider a generator standing to the right of a supernumber and commute it to the left of all quantum space coordinates. Proceeding in this manner one can verify a remarkable correspondence between right and left actions. More concretely, we have the transformations

$$f(\theta^1, \theta^2) \triangleleft T^\pm \xleftrightarrow{i \leftrightarrow i'} -q^{\mp 3} T^\mp \triangleright f(\theta^1, \theta^2), \quad (43)$$

$$f(\theta^1, \theta^2) \triangleleft \tau \xleftrightarrow{i \leftrightarrow i'} \tau \triangleright f(\theta^1, \theta^2),$$

where the symbol $\xleftrightarrow{i \leftrightarrow i'}$ indicates the following transitions:

$$\theta^i \xleftrightarrow{i \leftrightarrow i'} \theta^{i'}, \quad \theta^i \theta^j \xleftrightarrow{i \leftrightarrow i'} \theta^{j'} \theta^{i'}, \quad (44)$$

$$f' \xleftrightarrow{i \leftrightarrow i'} f', \quad f_i \xleftrightarrow{i \leftrightarrow i'} f_{i'}, \quad f_{ij} \xleftrightarrow{i \leftrightarrow i'} f_{j' i'},$$

$$i' \equiv 3 - i, \quad i, j = 1, 2.$$

For this to become clearer, we give as an example

$$\begin{aligned} -q^3 T^+ \triangleright f(\theta^1, \theta^2) &= -q^2 f_1 \theta^2 \xleftrightarrow{i \leftrightarrow i'} \\ -q^2 f_2 \theta^1 &= f(\theta^1, \theta^2) \triangleleft T^-. \end{aligned} \quad (45)$$

Now, we turn our attention to the covariant differential calculus on the quantum plane. The differentials of bosonival and fermionival coordinates are subject to the relations [cf. (20)]

$$\xi^i \xi^j = -q \hat{R}_{kl}^{ij} \xi^k \xi^l, \quad \eta^i \eta^j = q^{-1} \hat{R}_{kl}^{ij} \eta^k \eta^l, \quad (46)$$

which is consistent with [cf. (21) and (22)]

$$X^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k X^l, \quad \theta^i \eta^j = q^{-1} \hat{R}_{kl}^{ij} \eta^k \theta^l. \quad (47)$$

In what follows we need the q -deformed spinor metric ε^{ij} and its inverse ε_{ij} given by [42]

$$\varepsilon^{11} = \varepsilon^{22} = 0, \quad \varepsilon^{12} = -q^{-1/2}, \quad \varepsilon^{21} = q^{1/2}, \quad (48)$$

and

$$\varepsilon_{ij} = -\varepsilon^{ij}. \quad (49)$$

Now, we are able to raise and lower spinor indices as usual, i.e.

$$a_i = \varepsilon_{ij} a^j, \quad a^i = \varepsilon^{ij} a_j. \quad (50)$$

With the identity

$$\varepsilon^{nk} \hat{R}_{kl}^{ij} = q(\hat{R}^{-1})_{lk}^{ni} \varepsilon^{kj}, \quad (51)$$

one may check that for partial derivatives with upper indices the Leibniz rules in (24) become

$$\partial_x^i X^j = \varepsilon^{ij} + q^2 (\hat{R}^{-1})_{kl}^{ij} X^k \partial_x^l, \quad (52)$$

$$\partial_\theta^i \theta^j = \varepsilon^{ij} - (\hat{R}^{-1})_{kl}^{ij} \theta^k \partial_\theta^l.$$

Applying the substitutions

$$\partial_a \rightarrow \hat{\partial}_a, \quad a \rightarrow \bar{a}, \quad a \in \{\xi, \eta, X, \theta\}, \quad (53)$$

$$q \rightarrow q^{-1}, \quad \hat{R} \rightarrow \hat{R}^{-1},$$

to relations (46), (47), and (52) then yields the corresponding identities for the conjugated differential calculus.

Next, we want to deal on with the actions of partial derivatives on supernumbers. This way, we can proceed in very much the same way as was done in the case of

symmetry generators. Written out explicitly, the relations in (52) become for the fermionic case

$$\partial_\theta^1 \theta^1 = -q^{-1} \theta^1 \partial_\theta^1, \tag{54}$$

$$\partial_\theta^1 \theta^2 = -q^{-1/2} - \theta^2 \partial_\theta^1,$$

$$\partial_\theta^2 \theta^1 = q^{1/2} - \theta^1 \partial_\theta^2 + \lambda \theta^2 \partial_\theta^1, \tag{55}$$

$$\partial_\theta^2 \theta^2 = -q^{-1} \theta^2 \partial_\theta^2,$$

and likewise for the conjugated partial derivatives,

$$\hat{\partial}_\theta^1 \bar{\theta}^1 = -q \bar{\theta}^1 \hat{\partial}_\theta^1, \tag{56}$$

$$\hat{\partial}_\theta^1 \bar{\theta}^1 = -q^{-1/2} - \bar{\theta}^2 \hat{\partial}_\theta^1 - \lambda \bar{\theta}^1 \hat{\partial}_\theta^2,$$

$$\hat{\partial}_\theta^1 \bar{\theta}^1 = q^{1/2} - \bar{\theta}^1 \hat{\partial}_\theta^2, \tag{57}$$

$$\hat{\partial}_\theta^1 \bar{\theta}^1 = -q \bar{\theta}^2 \hat{\partial}_\theta^2,$$

where $\lambda = q - q^{-1}$. From (54) and (55) it follows that

$$\partial_\theta^1 \triangleright f(\theta^2, \theta^1) = -q^{-1/2} f_2 - q^{-1/2} f_{21} \theta^1, \tag{58}$$

$$\partial_\theta^2 \triangleright f(\theta^2, \theta^1) = q^{1/2} f_1 - q^{-1/2} f_{21} \theta^2.$$

Repeating the same steps as before for conjugated partial derivatives as well as right actions one can verify the correspondences

$$\hat{\partial}_\theta^i \bar{\triangleright} f(\bar{\theta}^1, \bar{\theta}^2) \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} -\partial_\theta^{i'} \triangleright f(\theta^2, \theta^1), \tag{59}$$

$$q^{-1} f(\bar{\theta}^2, \bar{\theta}^1) \bar{\triangleleft} \hat{\partial}_\theta^i \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} -q f(\theta^1, \theta^2) \triangleleft \partial_\theta^{i'},$$

and

$$f(\theta^1, \theta^2) \triangleleft \partial_\theta^i \xrightarrow{i \leftrightarrow i'} \overset{i \leftrightarrow i'}{\leftarrow} -q^{-1} \hat{\partial}_\theta^{i'} \bar{\triangleright} f(\bar{\theta}^1, \bar{\theta}^2), \tag{60}$$

$$f(\bar{\theta}^2, \bar{\theta}^1) \bar{\triangleleft} \hat{\partial}_\theta^i \xrightarrow{i \leftrightarrow i'} \overset{i \leftrightarrow i'}{\leftarrow} -q \partial_\theta^{i'} \triangleright f(\theta^2, \theta^1),$$

where the symbol $\xrightarrow{q \rightarrow 1/q}$ now describes substitutions given by

$$\theta^i \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} \theta^{i'}, \quad \theta^i \theta^j \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} \theta^{i'} \theta^{j'}, \tag{61}$$

$$f' \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} f', \quad f_i \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} f_{i'}, \quad f_{ij} \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} f_{i'j'},$$

$$q \xrightarrow{q \rightarrow 1/q} \overset{i \rightarrow i'}{\leftarrow} q^{-1}, \quad i, j = 1, 2.$$

It should be noticed that the normal ordering the representation of a supernumber refers to is indicated by the order in which arguments are arranged in the symbol for the supernumber (see also Appendix A).

Now, we come to the Hopf structure on quantum space generators. A brief glance at (30) and (31) shows us that the explicit form of coproduct and antipode is completely determined by the L -matrices. Therefore, our task is to find for the unknown entries of the L -matrix combinations

of symmetry generators that produce the correct commutation relations between generators of different quantum spaces. In other words, exploiting the identities

$$a^i b^j = ((\mathcal{L}_a)_k^i \triangleright b^j) a^k, \quad b^i a^j = ((\bar{\mathcal{L}}_b)_k^i \triangleright a^j) b^k, \tag{62}$$

we should be able to regain the relations in (46), (47), (52), and their conjugated counterparts (if inhomogeneous terms are discounted). We have found L -matrices with this property. Inserting their explicit form into (30) and (31) we get for coproduct, antipode, and counit the expressions

$$\Delta(a^1) = a^1 \otimes 1 + \Lambda(a) \tau^{-1/4} \otimes a^1, \tag{63}$$

$$\Delta(a^2) = a^2 \otimes 1 + \Lambda(a) \tau^{1/4} \otimes a^2 - q \lambda \Lambda(a) \tau^{-1/4} T^+ \otimes a^1,$$

$$S(a^1) = -\Lambda^{-1}(a) \tau^{1/4} a^1, \tag{64}$$

$$S(a^2) = -\Lambda^{-1}(a) \tau^{-1/4} a^2 - q^2 \lambda \Lambda^{-1}(a) \tau^{-1/4} T^+ a^1,$$

$$\varepsilon(a^1) = \varepsilon(a^2) = 0, \tag{65}$$

and similarly for the Hopf structure to the conjugated L -matrix,

$$\bar{\Delta}(a^1) = a^1 \otimes 1 + \Lambda^{-1}(a) \tau^{1/4} \otimes a^1 + q^{-1} \lambda \Lambda^{-1}(a) \tau^{-1/4} T^- \otimes a^2, \tag{66}$$

$$\bar{\Delta}(a^2) = a^2 \otimes 1 + \Lambda^{-1}(a) \tau^{-1/4} \otimes a^2,$$

$$\bar{S}(a^1) = -\Lambda(a) \tau^{-1/4} a^1 + q^{-2} \lambda \Lambda(a) \tau^{-1/4} T^- a^2, \tag{67}$$

$$\bar{S}(a^2) = -\Lambda(a) \tau^{1/4} a^2,$$

$$\bar{\varepsilon}(a^1) = \bar{\varepsilon}(a^2) = 0, \tag{68}$$

where a stands for one of the following quantities:

$$a \in \{\partial_x, \partial_\theta, X, \theta, \xi, \eta\}. \tag{69}$$

From (63) and (66) we can see that the L -matrices depend on unitary scaling operators denoted by $\Lambda(a)$. To understand the occurrence of these scaling operators we have to take a look at the commutation relations in (47) and (52), which tell us that the braiding between generators of different quantum spaces is given by the R -matrix or its inverse up to a constant factor. The point now is that the action of the scaling operators have been determined in such a way that the relations in (62) lead to the correct factors if we consider the braiding between generators of different quantum spaces. This can be achieved by specifying the scaling operators according to

$$\Lambda(\partial_x^i) = \Lambda^{-3/4}, \quad \Lambda(X^i) = \Lambda^{3/4}, \quad \Lambda(\eta^i) = \Lambda^{-1/4} \tag{70}$$

and

$$\Lambda(\partial_\theta^i) = \tilde{\Lambda}^{-3}, \quad \Lambda(\theta^i) = \tilde{\Lambda}^3, \quad \Lambda(\xi^i) = \tilde{\Lambda}^3, \tag{71}$$

where the grouplike operators Λ and $\tilde{\Lambda}$ satisfy the commutation relations

$$\Lambda X^i = q^{-2} X^i \Lambda, \quad \tilde{\Lambda} X^i = q^{3/2} X^i \tilde{\Lambda}, \tag{72}$$

$$\begin{aligned}
\Lambda \partial_x^i &= q^2 \partial_x^i \Lambda, & \tilde{\Lambda} \partial_x^i &= q^{-3/2} \partial_x^i \tilde{\Lambda}, \\
\Lambda \xi^i &= q^{-2} \xi^i \Lambda, & \tilde{\Lambda} \xi^i &= -q^{-1/2} \xi^i \tilde{\Lambda}, \\
\Lambda \eta^i &= q^{3/2} \eta^i \Lambda, & \tilde{\Lambda} \eta^i &= q^{-1/2} \eta^i \tilde{\Lambda}, \\
\Lambda \theta^i &= q^{-2} \theta^i \Lambda, & \tilde{\Lambda} \theta^i &= -q^{-1/2} \theta^i \tilde{\Lambda}, \\
\Lambda \partial_\theta^i &= q^2 \partial_\theta^i \Lambda, & \tilde{\Lambda} \partial_\theta^i &= -q^{1/2} \partial_\theta^i \tilde{\Lambda}.
\end{aligned}$$

Finally, let us notice that the above identities for the scaling operator have been derived by exploiting consistence arguments like

$$\begin{aligned}
a^i b^j &= ((\mathcal{L}_a)_k^i \triangleright b^j) a^k = b^k (a^i \triangleleft (\bar{\mathcal{L}}_b)_k^j), & (73) \\
b^i a^j &= ((\bar{\mathcal{L}}_b)_k^i \triangleright a^j) b^k = a^k (b^i \triangleleft (\mathcal{L}_a)_k^j).
\end{aligned}$$

4 Three-dimensional Euclidean space

All considerations of the previous sections pertain equally to the three-dimensional q -deformed Euclidean space [14]. Thus we can restrict ourselves to stating the results, only. Now, the projector decomposition of the R -matrix becomes

$$\hat{R} = P_S - q^{-4} P_A + q^{-6} P_T. \quad (74)$$

The relations for the fermionic quantum space coordinates are given by

$$\theta^A \theta^B = -q^4 \hat{R}_{CD}^{AB} \theta^C \theta^D, \quad (75)$$

which is equivalent to the following independent relations:

$$\begin{aligned}
(\theta^+)^2 &= (\theta^-)^2 = 0, & (76) \\
\theta^3 \theta^3 &= \lambda \theta^+ \theta^-, \\
\theta^+ \theta^- &= -\theta^- \theta^+, \\
\theta^\pm \theta^3 &= -q^{\pm 2} \theta^3 \theta^\pm.
\end{aligned}$$

Using the above relations one can show that the product of two supernumbers represented by

$$\begin{aligned}
&f(\theta^+, \theta^3, \theta^-) & (77) \\
&= f' + f_+ \theta^+ + f_3 \theta^3 + f_- \theta^- + f_{+3} \theta^+ \theta^3 \\
&\quad + f_{+-} \theta^+ \theta^- + f_{3-} \theta^3 \theta^- + f_{+3-} \theta^+ \theta^3 \theta^-
\end{aligned}$$

now becomes

$$\begin{aligned}
&(f \cdot g)(\theta^+, \theta^3, \theta^-) & (78) \\
&= (f \cdot g)' + (f \cdot g)_+ \theta^+ + (f \cdot g)_3 \theta^3 + (f \cdot g)_- \theta^- \\
&\quad + (f \cdot g)_{+3} \theta^+ \theta^3 + (f \cdot g)_{+-} \theta^+ \theta^- + (f \cdot g)_{3-} \theta^3 \theta^- \\
&\quad + (f \cdot g)_{+3-} \theta^+ \theta^3 \theta^-,
\end{aligned}$$

with

$$(f \cdot g)' = f' g', \quad (79)$$

$$(f \cdot g)_A = f_A g' + f' g_A, \quad A \in \{+, 3, -\},$$

$$(f \cdot g)_{+3} = f_+ g_3 - q^{-2} f_3 g_+ + f' g_{+3} + g' f_{+3}, \quad (80)$$

$$\begin{aligned}
&(f \cdot g)_{3-} = f_3 g_- - q^{-2} f_- g_3 + f' g_{3-} + g' f_{3-}, \\
&(f \cdot g)_{+-} = f_+ g_- - f_- g_+ + \lambda f_3 g_3 \\
&\quad + f' g_{+-} + g' f_{+-}, \\
&(f \cdot g)_{+3-} = f_+ g_{3-} - q^{-2} f_3 g_{+-} + q^{-2} f_- g_{+3} & (81) \\
&\quad + f_{+3} g_- - q^{-2} f_{+-} g_3 + q^{-2} f_{3-} g_+ \\
&\quad + f' g_{+3-} + g' f_{+3-}.
\end{aligned}$$

Next, we come to the action of symmetry generators on supernumbers. To this end, let us notice that fermionic coordinates of three-dimensional q -deformed Euclidean space transform under the action of $U_q(su_2)$ like the components of a vector. Thus, the commutation relations between the generators of $U_q(su_2)$ and the fermionic coordinates read

$$L^+ \theta^+ = \theta^+ L^+, \quad L^+ \theta^3 = \theta^3 L^+ - q \theta^+ \tau^{-1/2}, \quad (82)$$

$$L^+ \theta^- = \theta^- L^+ - \theta^3 \tau^{-1/2},$$

$$L^- \theta^+ = \theta^+ L^- + \theta^3 \tau^{-1/2},$$

$$L^- \theta^3 = \theta^3 L^- + q^{-1} \theta^- \tau^{-1/2}, \quad L^- \theta^- = \theta^- L^-, \quad (83)$$

$$\tau^{-1/2} \theta^\pm = q^{\pm 2} \theta^\pm \tau^{-1/2}, \quad \tau^{-1/2} \theta^3 = \theta^3 \tau^{-1/2}. \quad (84)$$

From these relations we get the representations

$$L^+ \triangleright f(\theta^+, \theta^3, \theta^-) \quad (85)$$

$$= -q f_3 \theta^+ - f_- \theta^3 - f_{+-} \theta^+ \theta^3 - q f_{3-} \theta^+ \theta^-,$$

$$L^- \triangleright f(\theta^+, \theta^3, \theta^-) \quad (86)$$

$$= q^{-1} f_3 \theta^- + f_+ \theta^3 + q f_{+3} \theta^+ \theta^- + q^{-2} f_{+-} \theta^3 \theta^-,$$

$$\tau^{-1/2} \triangleright f(\theta^+, \theta^3, \theta^-) = f(q^2 \theta^+, \theta^3, q^{-2} \theta^-). \quad (87)$$

They are related with right representations by either the transformation rules

$$f(\theta^+, \theta^3, \theta^-) \triangleleft L^\pm \overset{\pm \leftrightarrow \mp}{\longleftarrow} L^\mp \triangleright f(\theta^+, \theta^3, \theta^-), \quad (88)$$

or

$$f(\theta^+, \theta^3, \theta^-) \triangleleft \tau^{-1/2} = \tau^{1/2} \triangleright f(\theta^+, \theta^3, \theta^-), \quad (89)$$

The symbol $\overset{\pm \leftrightarrow \mp}{\longleftarrow}$ indicates the transitions

$$\theta^{A_1} \dots \theta^{A_n} \overset{\pm \leftrightarrow \mp}{\longleftarrow} \theta^{\bar{A}_n} \dots \theta^{\bar{A}_1}, \quad (90)$$

$$f_{A_1 \dots A_n} \overset{\pm \leftrightarrow \mp}{\longleftarrow} f_{\bar{A}_n \dots \bar{A}_1},$$

$$f' \overset{\pm \leftrightarrow \mp}{\longleftarrow} f',$$

where we have introduced indices with bar by $\bar{A} = (+, 3, -) = (-, 3, +)$.

Now, we turn to the differentials, which have to be subject to the relations

$$\xi^A \xi^B = -q^4 \hat{R}_{CD}^{AB} \xi^C \xi^D, \quad \eta^A \eta^B = \hat{R}_{CD}^{AB} \eta^C \eta^D, \quad (91)$$

and

$$X^A \xi^B = q^4 \hat{R}_{CD}^{AB} \xi^C X^D, \quad \theta^A \eta^B = \hat{R}_{CD}^{AB} \eta^C \theta^D. \quad (92)$$

The last two relations imply the Leibniz rules

$$\begin{aligned} \partial_x^A X^B &= g^{AB} + (\hat{R}^{-1})_{CD}^{AB} X^C \partial_x^D, \\ \partial_\theta^A \theta^B &= g^{AB} - q^{-4} (\hat{R}^{-1})_{CD}^{AB} \theta^C \partial_\theta^D, \end{aligned} \quad (93)$$

where g^{AB} denotes the quantum metric of the three-dimensional q -deformed Euclidean space. In complete analogy to the previous section, the relations for the conjugated quantities follow from the above identities by applying the substitutions

$$\begin{aligned} \partial_a &\rightarrow \hat{\partial}_a, \quad a \rightarrow \bar{a}, \quad a \in \{\xi, \eta, X, \theta\}, \\ q &\rightarrow q^{-1}, \quad \hat{R} \rightarrow \hat{R}^{-1}. \end{aligned} \quad (94)$$

For the fermionic derivatives the Leibniz rules read explicitly

$$\partial_\theta^+ \theta^+ = -q^{-4} \theta^+ \partial_\theta^+, \quad (95)$$

$$\partial_\theta^+ \theta^3 = -q^{-2} \theta^3 \partial_\theta^+ + q^{-2} \lambda \lambda_+ \theta^+ \partial_\theta^3,$$

$$\partial_\theta^+ \theta^- = -q - \theta^- \partial_\theta^+ + q^{-1} \lambda \lambda_+ \theta^3 \partial_\theta^3 - q^{-1} \lambda^2 \lambda_+ \theta^+ \partial_\theta^-,$$

$$\partial_\theta^3 \theta^+ = -q^{-2} \theta^+ \partial_\theta^3, \quad (96)$$

$$\partial_\theta^3 \theta^3 = 1 - q^{-2} \theta^3 \partial_\theta^3 + q^{-1} \lambda \lambda_+ \theta^+ \partial_\theta^-,$$

$$\partial_\theta^3 \theta^- = -q^{-2} \theta^- \partial_\theta^3 + q^{-2} \lambda \lambda_+ \theta^3 \partial_\theta^-,$$

$$\partial_\theta^- \theta^+ = -q^{-1} - \theta^+ \partial_\theta^-, \quad (97)$$

$$\partial_\theta^- \theta^3 = -q^{-2} \theta^3 \partial_\theta^-, \quad \partial_\theta^- \theta^- = -q^{-4} \theta^- \partial_\theta^-,$$

where $\lambda_+ \equiv q + q^{-1}$. By the substitutions

$$\partial_\theta^A \rightarrow \hat{\partial}_\theta^{\bar{A}}, \quad \theta^A \rightarrow \bar{\theta}^{\bar{A}}, \quad q \rightarrow q^{-1}, \quad (98)$$

we get the corresponding relations for the conjugated differential calculus. In a straightforward manner, we can derive from the identities in (95)–(97) the actions of the fermionic derivatives on supernumbers. This way, we obtain

$$\partial_\theta^+ \triangleright f(\theta^+, \theta^3, \theta^-) \quad (99)$$

$$= -q f_- + q^{-3} f_{+-} \theta^+ + q^{-1} f_{3-} \theta^3 - q^{-5} f_{+3-} \theta^+ \theta^3,$$

$$\partial_\theta^3 \triangleright f(\theta^+, \theta^3, \theta^-) \quad (100)$$

$$= f_3 - q^{-2} f_{+3} \theta^+ + f_{3-} \theta^- - q^{-2} f_{+3-} \theta^+ \theta^-,$$

$$\partial_\theta^- \triangleright f(\theta^+, \theta^3, \theta^-) \quad (101)$$

$$= -q^{-1} f_+ - q^{-1} f_{+3} \theta^3 - q^{-1} f_{+-} \theta^- - q^{-1} f_{+3-} \theta^3 \theta^-.$$

The relationship between the different types of representations is now given by

$$\partial_\theta^A \triangleright f(\theta^+, \theta^3, \theta^-) \xleftrightarrow{q \rightarrow 1/q} \hat{\partial}_\theta^{\bar{A}} \triangleright f(\bar{\theta}^-, \bar{\theta}^3, \bar{\theta}^+), \quad (102)$$

$$q^{-2} f(\bar{\theta}^+, \bar{\theta}^3, \bar{\theta}^-) \triangleleft \hat{\partial}_\theta^{\bar{A}} \xleftrightarrow{q \rightarrow 1/q} q^2 f(\theta^-, \theta^3, \theta^+) \triangleleft \partial_\theta^{\bar{A}},$$

and

$$f(\bar{\theta}^+, \bar{\theta}^3, \bar{\theta}^-) \triangleleft \hat{\partial}_\theta^{\bar{A}} \xleftrightarrow{q \rightarrow 1/q} q^2 \partial_\theta^{\bar{A}} \triangleright f(\theta^+, \theta^3, \theta^-), \quad (103)$$

$$f(\theta^-, \theta^3, \theta^+) \triangleleft \partial_\theta^{\bar{A}} \xleftrightarrow{q \rightarrow 1/q} q^{-2} \hat{\partial}_\theta^{\bar{A}} \triangleright f(\bar{\theta}^-, \bar{\theta}^3, \bar{\theta}^+),$$

where $\xleftrightarrow{q \rightarrow 1/q}$ denotes the transition given by

$$\theta^{A_1} \dots \theta^{A_n} \xleftrightarrow{q \rightarrow 1/q} \theta^{\bar{A}_1} \dots \theta^{\bar{A}_n}, \quad (104)$$

$$f_{A_1 \dots A_n} \xleftrightarrow{q \rightarrow 1/q} f_{\bar{A}_1 \dots \bar{A}_n},$$

$$f' \xleftrightarrow{q \rightarrow 1/q} f', \quad q \xleftrightarrow{q \rightarrow 1/q} q^{-1}.$$

Last but not least we would like to concentrate our attention to the Hopf structures for the various types of quantum spaces. In general we have

$$\Delta(a^-) = a^- \otimes 1 + \Lambda(a) \tau^{-1/2} \otimes a^-, \quad (105)$$

$$\Delta(a^3) = a^3 \otimes 1 + \Lambda(a) \otimes a^3 + \lambda \lambda_+ \Lambda(a) L^+ \otimes a^-,$$

$$\begin{aligned} \Delta(a^+) &= a^+ \otimes 1 + \Lambda(a) \tau^{1/2} \otimes a^+ \\ &+ q \lambda \lambda_+ \Lambda(a) \tau^{1/2} L^+ \otimes a^3 \\ &+ q^2 \lambda^2 \lambda_+ \Lambda(a) \tau^{1/2} (L^+)^2 \otimes a^-, \end{aligned}$$

$$S(a^-) = -\Lambda^{-1}(a) \tau^{1/2} a^-, \quad (106)$$

$$S(a^3) = -\Lambda^{-1}(a) a^3 + q^2 \lambda \lambda_+ \Lambda^{-1}(a) \tau^{1/2} L^+ a^-,$$

$$\begin{aligned} S(a^+) &= -\Lambda^{-1}(a) \tau^{-1/2} a^+ + q \lambda \lambda_+ \Lambda^{-1}(a) L^+ a^3 \\ &- q^4 \lambda^2 \lambda_+ \Lambda^{-1}(a) \tau^{1/2} (L^+)^2 a^-, \end{aligned}$$

$$\varepsilon(a^+) = \varepsilon(a^3) = \varepsilon(a^-) = 0, \quad (107)$$

and likewise in the conjugated case

$$\bar{\Delta}(a^+) = a^+ \otimes 1 + \Lambda^{-1}(a) \tau^{-1/2} \otimes a^+, \quad (108)$$

$$\begin{aligned} \bar{\Delta}(a^3) &= a^3 \otimes 1 + \Lambda^{-1}(a) \otimes a^3 \\ &+ \lambda \lambda_+ \Lambda^{-1}(a) L^- \otimes a^+, \end{aligned}$$

$$\begin{aligned} \bar{\Delta}(a^-) &= a^- \otimes 1 + \Lambda^{-1}(a) \tau^{-1/2} \otimes a^- \\ &+ q^{-1} \lambda \lambda_+ \Lambda^{-1}(a) \tau^{1/2} L^- \otimes a^3 \\ &+ q^{-2} \lambda^2 \lambda_+ \Lambda^{-1}(a) \tau^{1/2} (L^-)^2 \otimes a^+, \end{aligned}$$

$$\bar{S}(a^+) = -\Lambda(a) \tau^{1/2} a^+, \quad (109)$$

$$\bar{S}(a^3) = -\Lambda(a) a^3 + q^{-2} \lambda \lambda_+ \Lambda(a) \tau^{1/2} L^- a^+,$$

$$\begin{aligned} \bar{S}(a^-) &= -\Lambda(a) \tau^{-1/2} a^- + q^{-1} \lambda \lambda_+ \Lambda(a) L^- a^3 \\ &- q^{-4} \lambda^2 \lambda_+ \Lambda(a) \tau^{1/2} (L^-)^2 a^+, \end{aligned}$$

$$\bar{\varepsilon}(a^+) = \bar{\varepsilon}(a^3) = \bar{\varepsilon}(a^-) = 0, \quad (110)$$

where a again denotes one of the following objects:

$$a \in \{\partial_x, \partial_\theta, X, \theta, \xi, \eta\}. \quad (111)$$

The scaling operators have to be specified by

$$\Lambda(\partial_x^A) = \Lambda^{1/2}, \quad \Lambda(X^A) = \Lambda^{-1/2}, \quad \Lambda(\eta^A) = \Lambda^{1/2}, \quad (112)$$

and

$$\Lambda(\partial_\theta^A) = \tilde{\Lambda}, \quad \Lambda(\theta^A) = \tilde{\Lambda}^{-1}, \quad \Lambda(\xi^A) = \tilde{\Lambda}^{-1}, \quad (113)$$

which requires the operators Λ and $\tilde{\Lambda}$ to satisfy

$$\begin{aligned} \Lambda X^A &= q^4 X^A \Lambda, & \tilde{\Lambda} X^A &= q^2 X^A \tilde{\Lambda}, \\ \Lambda \partial_x^A &= q^{-4} \partial_x^A \Lambda, & \tilde{\Lambda} \partial_x^A &= q^{-2} \partial_x^A \tilde{\Lambda}, \\ \Lambda \xi^A &= q^4 \xi^A \Lambda, & \tilde{\Lambda} \xi^A &= -q^{-2} \xi^A \tilde{\Lambda}, \\ \Lambda \eta^A &= q^{-4} \eta^A \Lambda, & \tilde{\Lambda} \eta^A &= q^{-2} \eta^A \tilde{\Lambda}, \\ \Lambda \theta^A &= q^4 \theta^A \Lambda, & \tilde{\Lambda} \theta^A &= -q^{-2} \theta^A \tilde{\Lambda}, \\ \Lambda \partial_\theta^A &= q^{-4} \partial_\theta^A \Lambda, & \tilde{\Lambda} \partial_\theta^A &= -q^2 \partial_\theta^A \tilde{\Lambda}. \end{aligned} \quad (114)$$

5 Four-dimensional Euclidean space

The q -deformed Euclidean space in four dimensions can be treated in very much the same way as the Euclidean space in three dimensions. Thus, we summarize our results only. The projector decomposition for the R -matrix is [32, 43]

$$\hat{R}^{-1} = q^{-1} P_S - q P_A + q^3 P_T. \quad (115)$$

The commutation relations among the fermionic coordinates can be written in the general form

$$\theta^i \theta^j = -q \hat{R}_{kl}^{ij} \theta^k \theta^l, \quad (116)$$

which leads to the independent relations

$$\begin{aligned} (\theta^i)^2 &= 0, \quad i = 1, \dots, 4, \\ \theta^1 \theta^j &= -q^{-1} \theta^j \theta^1, \quad j = 1, 2, \\ \theta^j \theta^4 &= -q^{-1} \theta^4 \theta^j, \\ \theta^1 \theta^4 &= -\theta^4 \theta^1, \quad \theta^2 \theta^3 = -\theta^3 \theta^2 + \lambda \theta^1 \theta^4. \end{aligned} \quad (117)$$

For supernumbers of the form

$$\begin{aligned} f(\theta^1, \theta^2, \theta^3, \theta^4) & \\ &= f' + \sum_{i=1}^4 f_i \theta^i + \sum_{1 \leq i_1 < i_2 \leq 4} f_{i_1 i_2} \theta^{i_1} \theta^{i_2} \\ &+ \sum_{1 \leq i_1 < i_2 < i_3 \leq 4} f_{i_1 i_2 i_3} \theta^{i_1} \theta^{i_2} \theta^{i_3} + f_{1234} \theta^1 \theta^2 \theta^3 \theta^4, \end{aligned} \quad (118)$$

we can again calculate an expression for their product. Explicitly, we have

$$\begin{aligned} (f \cdot g)(\theta^1, \theta^2, \theta^3, \theta^4) & \\ &= (f \cdot g)' + \sum_{i=1}^4 (f \cdot g)_i \theta^i + \sum_{1 \leq i_1 < i_2 \leq 4} (f \cdot g)_{i_1 i_2} \theta^{i_1} \theta^{i_2} \\ &+ \sum_{1 \leq i_1 < i_2 < i_3 \leq 4} (f \cdot g)_{i_1 i_2 i_3} \theta^{i_1} \theta^{i_2} \theta^{i_3} + (f \cdot g)_{1234} \theta^1 \theta^2 \theta^3 \theta^4, \end{aligned} \quad (119)$$

with

$$(f \cdot g)' = f' g', \quad (120)$$

$$(f \cdot g)_i = f_i g' + f' g_i, \quad i = 1, \dots, 4,$$

$$(f \cdot g)_{1j} = f_{1j} g' + f' g_{1j} + f_1 g_j - q f_j g_1, \quad j = 2, 3,$$

$$(f \cdot g)_{j4} = f_{j4} g' + f' g_{j4} + f_j g_4 - q f_4 g_j, \quad (121)$$

$$(f \cdot g)_{23} = f_{23} g' + f' g_{23} + f_2 g_3 - f_3 g_2,$$

$$(f \cdot g)_{14} = f_{14} g' + f' g_{14} + f_1 g_4 - f_4 g_1 + \lambda f_3 g_2,$$

$$\begin{aligned} (f \cdot g)_{123} &= f_{123} g' + f' g_{123} + f_1 g_{23} - q f_2 g_{13} + q f_3 g_{12} \\ &+ f_{12} g_3 - f_{13} g_2 + q^2 f_{23} g_1, \end{aligned} \quad (122)$$

$$\begin{aligned} (f \cdot g)_{124} &= f_{124} g' + f' g_{124} + f_1 g_{24} - q f_2 g_{14} + q f_4 g_{12} \\ &+ f_{12} g_4 - q f_{14} g_2 - q \lambda f_{23} g_2 + q f_{24} g_1, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{134} &= f_{134} g' + f' g_{134} + f_1 g_{34} + q f_3 g_{14} \\ &- q \lambda f_3 g_{23} + q f_4 g_{13} + f_{13} g_4 - q f_{14} g_3 \\ &+ q f_{34} g_1, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{234} &= f_{234} g' + f' g_{234} + f_2 g_{34} - f_3 g_{24} \\ &+ q^2 f_4 g_{23} + f_{23} g_4 - q f_{24} g_3 + q f_{34} g_2, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{1234} &= f_{1234} g' + f' g_{1234} + f_1 g_{234} - q f_2 g_{134} \\ &+ q f_3 g_{124} - q^2 f_4 g_{123} + f_{12} g_{34} - f_{13} g_{24} \\ &+ q^2 f_{14} g_{23} + q^2 f_{23} g_{14} - q^2 \lambda f_{23} g_{23} \\ &- q^2 f_{24} g_{13} + q^2 f_{34} g_{12} + f_{123} g_4 - q f_{124} g_3 \\ &+ q f_{134} g_2 - q^2 f_{234} g_1. \end{aligned} \quad (123)$$

Next, we come to the commutation relations between symmetry generators of $U_q(so_4)$ (we use here the form as it was presented in [43]) and fermionic coordinates:

$$\begin{aligned} L_1^+ \theta^1 &= q \theta^1 L_1^+ - q^{-1} \theta^2, & L_1^+ \theta^2 &= q^{-1} \theta^2 L_1^+, \\ L_1^+ \theta^3 &= q \theta^3 L_1^+ + q^{-1} \theta^4, & L_1^+ \theta^4 &= q^{-1} \theta^4 L_1^+, \end{aligned} \quad (124)$$

$$\begin{aligned} L_2^+ \theta^1 &= q \theta^1 L_2^+ - q^{-1} \theta^3, & L_2^+ \theta^2 &= q \theta^2 L_2^+ + q^{-1} \theta^4, \\ L_2^+ \theta^3 &= q^{-1} \theta^3 L_2^+, & L_2^+ \theta^4 &= q^{-1} \theta^4 L_2^+, \end{aligned} \quad (125)$$

$$\begin{aligned} L_1^- \theta^1 &= q \theta^1 L_1^-, & L_1^- \theta^2 &= q^{-1} \theta^2 L_1^- - q \theta^1, \\ L_1^- \theta^3 &= q \theta^3 L_1^-, & L_1^- \theta^4 &= q^{-1} \theta^4 L_1^- + q \theta^3, \end{aligned} \quad (126)$$

$$L_2^- \theta^1 = q \theta^1 L_2^-, \quad L_2^- \theta^2 = q \theta^2 L_2^-, \quad (127)$$

$$L_2^- \theta^3 = q^{-1} \theta^3 L_2^- - q \theta^1, \quad L_2^- \theta^4 = q^{-1} \theta^4 L_2^- + q \theta^2,$$

$$K_1 \theta^1 = q^{-1} \theta^1 K_1, \quad K_1 \theta^2 = q \theta^2 K_1, \quad (128)$$

$$K_1 \theta^3 = q^{-1} \theta^3 K_1, \quad K_1 \theta^4 = q \theta^4 K_1,$$

$$K_2 \theta^1 = q^{-1} \theta^1 K_2, \quad K_2 \theta^2 = q^{-1} \theta^2 K_2,$$

$$K_2 \theta^3 = q \theta^3 K_2, \quad K_2 \theta^4 = q \theta^4 K_2. \quad (129)$$

With these relations it is straightforward to show that the actions of the symmetry generators on supernumbers take

the form

$$L_1^+ \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \quad (130)$$

$$\begin{aligned} &= -q^{-1}f_1\theta^2 + q^{-1}f_3\theta^4 + f_{13}\theta^1\theta^4 - q^{-1}f_{13}\theta^2\theta^3 \\ &\quad + (q^{-2}f_{23} - q^{-1}f_{14})\theta^2\theta^4 \\ &\quad + q^{-1}f_{123}\theta^1\theta^2\theta^4 - q^{-1}f_{134}\theta^2\theta^3\theta^4, \end{aligned}$$

$$L_2^+ \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \quad (131)$$

$$\begin{aligned} &= -q^{-1}f_1\theta^3 + q^{-1}f_2\theta^4 + q^{-2}f_{12}\theta^1\theta^4 + q^1f_{12}\theta^2\theta^3 \\ &\quad - (q^{-1}f_{14} + f_{23})\theta^3\theta^4 \\ &\quad - q^{-1}f_{123}\theta^1\theta^3\theta^4 + q^{-1}f_{124}\theta^2\theta^3\theta^4, \end{aligned}$$

$$L_1^- \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \quad (132)$$

$$\begin{aligned} &= -qf_2\theta^1 + qf_4\theta^3 - qf_{24}\theta^1\theta^4 + f_{24}\theta^2\theta^3 \\ &\quad + q(qf_{14} - f_{23})\theta^1\theta^3 \\ &\quad + qf_{124}\theta^1\theta^2\theta^3 - qf_{234}\theta^1\theta^3\theta^4, \end{aligned}$$

$$L_2^- \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \quad (133)$$

$$\begin{aligned} &= -qf_3\theta^1 + qf_4\theta^2 - q^{-1}f_{34}\theta^1\theta^4 - f_{34}\theta^2\theta^3 \\ &\quad + q^2(f_{14} + qf_{23})\theta^1\theta^2 \\ &\quad - qf_{134}\theta^1\theta^2\theta^3 + qf_{234}\theta^1\theta^2\theta^4, \end{aligned}$$

and

$$\begin{aligned} K_1 &\triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \\ &= f(q^{-1}\theta^1, q\theta^2, q^{-1}\theta^3, q\theta^4), \\ K_2 &\triangleright f(\theta^1, \theta^2, \theta^3, \theta^4) \\ &= f(q^{-1}\theta^1, q^{-1}\theta^2, q\theta^3, q\theta^4). \end{aligned} \quad (134)$$

If we are interested in right representations, we can either apply the transformation rules

$$f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft L_i^\pm \xleftrightarrow{i \leftrightarrow i'} q^{\mp 3} L_i^\mp \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4), \quad (135)$$

or

$$\begin{aligned} f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft K_1 &= K_1^{-1} \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4), \\ f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft K_2 &= K_2^{-1} \triangleright f(\theta^1, \theta^2, \theta^3, \theta^4), \end{aligned} \quad (136)$$

where $\xleftrightarrow{i \leftrightarrow i'}$ denotes the transition

$$\begin{aligned} \theta^{i_1} \dots \theta^{i_n} &\xleftrightarrow{i \leftrightarrow i'} \theta^{i'_1} \dots \theta^{i'_n}, \\ f_{i_1 \dots i_n} &\xleftrightarrow{i \leftrightarrow i'} f_{i'_1 \dots i'_n}, \\ f' &\xleftrightarrow{i \leftrightarrow i'} f', \end{aligned} \quad (137)$$

and the conjugated index is given by $i' \equiv 5 - i$.

For the differentials we know that the relations [36]

$$\xi^i \xi^j = -q \hat{R}_{kl}^{ij} \xi^k \xi^l, \quad \eta^i \eta^j = q^{-1} \hat{R}_{kl}^{ij} \eta^i \eta^k \quad (138)$$

and

$$X^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k X^l, \quad \theta^i \eta^j = q^{-1} \hat{R}_{kl}^{ij} \eta^i \theta^j \quad (139)$$

hold. Using these identities we can verify that the Leibniz rules now take the form

$$\begin{aligned} \partial_x^i X^j &= g^{ij} + q(\hat{R}^{-1})_{kl}^{ij} X^k \partial_x^l, \\ \partial_\theta^i \theta^j &= g^{ij} - q^{-1}(\hat{R}^{-1})_{kl}^{ij} \theta^k \partial_\theta^l, \end{aligned} \quad (140)$$

where g^{ij} denotes the four-dimensional quantum space metric. Again, the relations of the conjugated differential calculus are obtained most easily by applying the substitutions

$$\begin{aligned} \partial_a &\rightarrow \hat{\partial}_a, \quad a \rightarrow \bar{a}, \quad a \in \{\xi, \eta, X, \theta\}, \\ q &\rightarrow q^{-1}, \quad \hat{R} \rightarrow \hat{R}^{-1}, \end{aligned} \quad (141)$$

Written out explicitly, the Leibniz rules become in the fermionic case

$$\partial_\theta^1 \theta^1 = -q^{-2} \theta^1 \partial_\theta^1, \quad \partial_\theta^1 \theta^2 = -q^{-1} \theta^2 \partial_\theta^1, \quad (142)$$

$$\partial_\theta^1 \theta^3 = -q^{-1} \theta^3 \partial_\theta^1, \quad \partial_\theta^1 \theta^4 = q^{-1} - \theta^4 \partial_\theta^1,$$

$$\partial_\theta^2 \theta^1 = -q^{-1} \theta^1 \partial_\theta^2 + q^{-1} \lambda \theta^2 \partial_\theta^1, \quad \partial_\theta^2 \theta^2 = -q^{-2} \theta^2 \partial_\theta^2,$$

$$\partial_\theta^2 \theta^3 = 1 - \theta^3 \partial_\theta^2 - \lambda \theta^4 \partial_\theta^1, \quad \partial_\theta^2 \theta^4 = -q^{-1} \theta^4 \partial_\theta^2, \quad (143)$$

$$\partial_\theta^3 \theta^1 = -q^{-1} \theta^1 \partial_\theta^3 + q^{-1} \lambda \theta^3 \partial_\theta^1, \quad (144)$$

$$\partial_\theta^3 \theta^2 = 1 - \theta^2 \partial_\theta^3 - \lambda \theta^4 \partial_\theta^1, \quad \partial_\theta^3 \theta^3 = -q^{-2} \theta^3 \partial_\theta^3,$$

$$\partial_\theta^3 \theta^4 = -q^{-1} \theta^4 \partial_\theta^3,$$

$$\partial_\theta^4 \theta^1 = q - \theta^1 \partial_\theta^4 - \lambda(\theta^2 \partial_\theta^3 + \theta^3 \partial_\theta^2 + \lambda \theta^4 \partial_\theta^1), \quad (145)$$

$$\partial_\theta^4 \theta^2 = -q^{-1} \theta^2 \partial_\theta^4 + q^{-1} \lambda \theta^4 \partial_\theta^2,$$

$$\partial_\theta^4 \theta^3 = -q^{-1} \theta^3 \partial_\theta^4 + q^{-1} \lambda \theta^4 \partial_\theta^3, \quad \partial_\theta^4 \theta^4 = -q^{-2} \theta^4 \partial_\theta^4,$$

while the substitutions

$$\partial_\theta^i \rightarrow \hat{\partial}_\theta^{i'}, \quad \theta^i \rightarrow \bar{\theta}^{i'}, \quad q \rightarrow q^{-1}, \quad (146)$$

lead to the corresponding relations for the conjugated differential calculus. With the same reasonings already applied in the previous sections we find

$$\partial_\theta^1 \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) \quad (147)$$

$$\begin{aligned} &= q^{-1}f_4 + q^{-1}f_{41}\theta^1 + q^{-1}f_{42}\theta^2 + q^{-1}f_{43}\theta^3 \\ &\quad + q^{-1}f_{421}\theta^2\theta^1 + q^{-1}f_{432}\theta^3\theta^2 + q^{-1}f_{431}\theta^3\theta^1 \\ &\quad + q^{-1}f_{4321}\theta^3\theta^2\theta^1, \end{aligned}$$

$$\partial_\theta^2 \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) \quad (148)$$

$$\begin{aligned} &= f_3 + f_{31}\theta^1 + f_{32}\theta^2 - q^{-1}f_{43}\theta^4 + f_{321}\theta^2\theta^1 \\ &\quad - q^{-1}f_{431}\theta^4\theta^1 - q^{-1}f_{432}\theta^4\theta^2 - q^{-1}f_{4321}\theta^4\theta^2\theta^1, \end{aligned}$$

$$\partial_\theta^3 \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) \quad (149)$$

$$\begin{aligned} &= f_2 + f_{21}\theta^1 - q^{-2}f_{32}\theta^3 - q^{-1}f_{42}\theta^4 - q^{-2}f_{321}\theta^3\theta^1 \\ &\quad - q^{-1}f_{421}\theta^4\theta^1 + q^{-3}f_{432}\theta^4\theta^3 + q^{-3}f_{4321}\theta^4\theta^3\theta^1, \end{aligned}$$

$$\partial_\theta^4 \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) \quad (150)$$

$$\begin{aligned} &= qf_1 - f_{21}\theta^2 - f_{31}\theta^3 - q^{-1}(f_{41} - \lambda f_{32})\theta^4 \\ &\quad + q^{-1}f_{321}\theta^3\theta^2 + q^{-2}f_{421}\theta^4\theta^2 + q^{-2}f_{431}\theta^4\theta^3 \end{aligned}$$

$$+q^{-1}\lambda f_{321}\theta^4\theta^1 - q^{-3}f_{4321}\theta^4\theta^3\theta^2.$$

The different types of representations are linked via

$$\partial_\theta^i \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1) \quad (151)$$

$$\begin{aligned} & \xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} \hat{\partial}_\theta^{i'} \triangleright f(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4), \\ & q^{-2}f(\bar{\theta}^4, \bar{\theta}^3, \bar{\theta}^2, \bar{\theta}^1) \triangleleft \hat{\partial}_\theta^i \end{aligned} \quad (152)$$

$$\xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} q^2 f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft \partial_\theta^{i'}$$

and

$$f(\bar{\theta}^4, \bar{\theta}^3, \bar{\theta}^2, \bar{\theta}^1) \triangleleft \hat{\partial}_\theta^{i'} \quad (153)$$

$$\begin{aligned} & \xleftrightarrow{i \rightarrow i'} q^2 \partial_\theta^i \triangleright f(\theta^4, \theta^3, \theta^2, \theta^1), \\ & f(\theta^1, \theta^2, \theta^3, \theta^4) \triangleleft \partial_\theta^{i'} \end{aligned} \quad (154)$$

$$\xleftrightarrow{i \rightarrow i'} q^{-2} \hat{\partial}_\theta^i \triangleright f(\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3, \bar{\theta}^4),$$

where $\xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'}$ stands for

$$\theta^{i_1} \dots \theta^{i_n} \xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} \theta^{i'_1} \dots \theta^{i'_n}, \quad (155)$$

$$f_{i_1 \dots i_n} \xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} f_{i'_1 \dots i'_n},$$

$$f' \xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} f',$$

$$q \xleftrightarrow[q \rightarrow 1/q]{i \rightarrow i'} q^{-1}.$$

Finally, we would like to present the Hopf structures for the various four-dimensional quantum spaces. In general, we have

$$\Delta(a^1) = a^1 \otimes 1 + \Lambda(a)K_1^{1/2}K_2^{1/2} \otimes a^1, \quad (156)$$

$$\Delta(a^2) = a^2 \otimes 1 + \Lambda(a)K_1^{-1/2}K_2^{1/2} \otimes a^2$$

$$+q\lambda\Lambda(a)K_1^{1/2}K_2^{1/2}L_1^+ \otimes a^1,$$

$$\Delta(a^3) = a^3 \otimes 1 + \Lambda(a)K_1^{1/2}K_2^{-1/2} \otimes a^3$$

$$+q\lambda\Lambda(a)K_1^{1/2}K_2^{1/2}L_2^+ \otimes a^1,$$

$$\Delta(a^4) = a^4 \otimes 1 + \Lambda(a)K_1^{-1/2}K_2^{-1/2} \otimes a^4$$

$$-q^2\lambda^2\Lambda(a)K_1^{1/2}K_2^{1/2}L_1^+L_2^+ \otimes a^1$$

$$-q\lambda\Lambda(a)K_1^{-1/2}K_2^{1/2}L_2^+ \otimes a^2$$

$$-q\lambda\Lambda(a)K_1^{1/2}K_2^{-1/2}L_1^+ \otimes a^3,$$

$$S(a^1) = -\Lambda^{-1}(a)K_1^{-1/2}K_2^{-1/2}a^1, \quad (157)$$

$$S(a^2) = -\Lambda^{-1}(a)K_1^{1/2}K_2^{-1/2}(a^2 - q^2\lambda L_1^+a^1),$$

$$S(a^3) = -\Lambda^{-1}(a)K_1^{-1/2}K_2^{1/2}(a^3 - q^2\lambda L_2^+a^1),$$

$$S(a^4) = -\Lambda^{-1}(a)K_1^{1/2}K_2^{1/2}(a^4 + q^2\lambda(L_1^+a^3 + L_2^+a^2))$$

$$-q^4\lambda^2\Lambda^{-1}(a)K_1^{1/2}K_2^{1/2}L_1^+L_2^+a^1,$$

$$\varepsilon(a^1) = \varepsilon(a^2) = \varepsilon(a^3) = \varepsilon(a^4) = 0, \quad (158)$$

and

$$\bar{\Delta}(a^1) = a^1 \otimes 1 + \Lambda^{-1}(a)K_1^{-1/2}K_2^{-1/2} \otimes a^1 \quad (159)$$

$$-q^{-2}\lambda^2\Lambda^{-1}(a)K_1^{1/2}K_2^{1/2}L_1^-L_2^- \otimes a^4$$

$$-q^{-1}\lambda\Lambda^{-1}(a)K_1^{1/2}K_2^{-1/2}L_1^- \otimes a^2$$

$$-q^{-1}\lambda\Lambda^{-1}(a)K_1^{-1/2}K_2^{1/2}L_2^- \otimes a^3,$$

$$\bar{\Delta}(a^2) = a^2 \otimes 1 + \Lambda^{-1}(a)K_1^{1/2}K_2^{-1/2} \otimes a^2$$

$$-q^{-1}\lambda\Lambda^{-1}(a)K_1^{1/2}K_2^{1/2}L_2^- \otimes a^4,$$

$$\bar{\Delta}(a^3) = a^3 \otimes 1 + \Lambda^{-1}(a)K_1^{-1/2}K_2^{1/2} \otimes a^3$$

$$+q^{-1}\lambda\Lambda^{-1}(a)K_1^{1/2}K_2^{1/2}L_1^- \otimes a^4,$$

$$\bar{\Delta}(a^4) = a^4 \otimes 1 + \Lambda^{-1}(a)K_1^{1/2}K_2^{1/2} \otimes a^4,$$

$$\bar{S}(a^1) = -\Lambda(a)K_1^{1/2}K_2^{1/2}(a^1 + q^{-2}\lambda(L_1^-a^2 + L_2^-a^3))$$

$$+q^{-4}\lambda^2\Lambda(a)K_1^{1/2}K_2^{1/2}L_1^-L_2^-a^4, \quad (160)$$

$$\bar{S}(a^2) = -\Lambda(a)K_1^{-1/2}K_2^{1/2}(a^2 - q^{-2}\lambda L_2^-a^4),$$

$$\bar{S}(a^3) = -\Lambda(a)K_1^{-1/2}K_2^{1/2}(a^3 - q^{-2}\lambda L_1^-a^4),$$

$$\bar{S}(a^4) = -\Lambda(a)K_1^{-1/2}K_2^{-1/2}a^4,$$

$$\bar{\varepsilon}(a^1) = \bar{\varepsilon}(a^2) = \bar{\varepsilon}(a^3) = \bar{\varepsilon}(a^4) = 0, \quad (161)$$

where

$$a \in \{\partial_x, \partial_\theta, X, \theta, \xi, \eta\}. \quad (162)$$

In order to regain relations (147)–(150) and their conjugated versions from the L -matrices determining the coproducts in (156) and (159), we have to represent the operators $\Lambda(a)$ as

$$\Lambda(\partial_x^i) = \Lambda^{1/2}, \quad \Lambda(X^i) = \Lambda^{-1/2}, \quad \Lambda(\eta^i) = \Lambda^{1/2} \quad (163)$$

and

$$\Lambda(\partial_\theta^i) = \tilde{\Lambda}^{-1}, \quad \Lambda(\theta^i) = \tilde{\Lambda}, \quad \Lambda(\xi^i) = \tilde{\Lambda}, \quad (164)$$

which requires one to impose on the unitary and grouplike scaling operators Λ and $\tilde{\Lambda}$ the commutation relations

$$\Lambda X^i = q^2 X^i \Lambda, \quad \tilde{\Lambda} X^i = q^{-1} X^i \tilde{\Lambda}, \quad \Lambda \partial_x^i = q^{-2} \partial_x^i \Lambda,$$

$$\tilde{\Lambda} \partial_x^i = q \partial_x^i \tilde{\Lambda}, \quad \Lambda \xi^i = q^2 \xi^i \Lambda, \quad \tilde{\Lambda} \xi^i = -q \xi^i \tilde{\Lambda}, \quad (165)$$

$$\Lambda \eta^i = q^2 \eta^i \Lambda, \quad \tilde{\Lambda} \eta^i = q \eta^i \tilde{\Lambda}, \quad \Lambda \theta^i = q^2 \theta^i \Lambda,$$

$$\tilde{\Lambda} \theta^i = -q \theta^i \tilde{\Lambda}, \quad \Lambda \partial_\theta^i = q^{-2} \partial_\theta^i \Lambda, \quad \tilde{\Lambda} \partial_\theta^i = -q^{-1} \partial_\theta^i \tilde{\Lambda}.$$

6 Minkowski space

In this section we would like to deal with q -deformed Minkowski space [10, 12–14, 44] which from a physical point

of view is the most interesting case in this article (for other deformations of spacetime and their related symmetries we refer to [45–50]). We follow the same line of arguments as in the previous sections. The R -matrix now obeys the decomposition [33]

$$\hat{R}_{II} = q^{-2}P_S - P_A + q^2P_T. \quad (166)$$

The relations for the fermionic coordinates are completely determined by

$$\theta^i\theta^j = -(\hat{R}_{II})_{kl}^{ij}\theta^k\theta^l, \quad (167)$$

from which we obtain as independent relations

$$\begin{aligned} (\theta^\mu)^2 &= 0, \quad \mu \in \{+, -, 0\}, \quad (168) \\ \theta^3\theta^\pm &= -q^{\mp 2}\theta^\pm\theta^3, \\ \theta^3\theta^3 &= \lambda\theta^+\theta^-, \quad \theta^+\theta^- = -\theta^-\theta^+, \\ \theta^\pm\theta^0 + \theta^0\theta^\pm &= \pm q^{\mp 1}\lambda\theta^\pm\theta^3, \\ \theta^0\theta^3 + \theta^3\theta^0 &= \lambda\theta^+\theta^-. \end{aligned}$$

Instead of dealing with the coordinate θ^3 or θ^0 it is often more convenient to work with the light-cone coordinate $\theta^{3/0} = \theta^3 - \theta^0$, for which we have the additional relations

$$\begin{aligned} (\theta^{3/0})^2 &= 0, \quad \theta^\pm\theta^{3/0} = -\theta^{3/0}\theta^\pm, \quad (169) \\ \theta^0\theta^{3/0} + \theta^{3/0}\theta^0 &= -\lambda\theta^+\theta^-, \\ \theta^\pm\theta^0 + q^{\pm 2}\theta^0\theta^\pm &= \pm q^{\pm 1}\lambda\theta^\pm\theta^{3/0}, \\ \theta^3\theta^{3/0} + \theta^{3/0}\theta^3 &= -\lambda\theta^+\theta^-. \end{aligned}$$

The product of two supernumbers of the form

$$\begin{aligned} f(\theta^+, \theta^3, \theta^0, \theta^-) \quad (170) \\ = f' + f_+\theta^+ + f_0\theta^0 + f_3\theta^3 + f_-\theta^- + f_{+3}\theta^+\theta^3 \\ + f_{+0}\theta^+\theta^0 + f_{+-}\theta^+\theta^- + f_{30}\theta^3\theta^0 + f_{3-}\theta^3\theta^- \\ + f_{0-}\theta^0\theta^- + f_{+30}\theta^+\theta^3\theta^0 + f_{+3-}\theta^+\theta^3\theta^- \\ + f_{+0-}\theta^+\theta^0\theta^- + f_{30-}\theta^3\theta^0\theta^- + f_{+30-}\theta^+\theta^3\theta^0\theta^- \end{aligned}$$

now becomes

$$\begin{aligned} (f \cdot g)(\theta^+, \theta^3, \theta^0, \theta^-) \quad (171) \\ = (f \cdot g)' + (f \cdot g)_+\theta^+ + (f \cdot g)_0\theta^0 + (f \cdot g)_3\theta^3 \\ + (f \cdot g)_-\theta^- + (f \cdot g)_{+3}\theta^+\theta^3 + (f \cdot g)_{+0}\theta^+\theta^0 \\ + (f \cdot g)_{+-}\theta^+\theta^- + (f \cdot g)_{30}\theta^3\theta^0 + (f \cdot g)_{3-}\theta^3\theta^- \\ + (f \cdot g)_{0-}\theta^0\theta^- + (f \cdot g)_{+30}\theta^+\theta^3\theta^0 \\ + (f \cdot g)_{+3-}\theta^+\theta^3\theta^- + (f \cdot g)_{+0-}\theta^+\theta^0\theta^- \\ + (f \cdot g)_{30-}\theta^3\theta^0\theta^- + (f \cdot g)_{+30-}\theta^+\theta^3\theta^0\theta^-, \end{aligned}$$

with

$$\begin{aligned} (f \cdot g)' &= f'g', \quad (172) \\ (f \cdot g)_\mu &= f_\mu g' + f'g_\mu, \quad \mu \in \{+, 3, 0, -\}, \end{aligned}$$

$$(f \cdot g)_{+0} = f_{+0}g' + f'g_{+0} + f_+g_0 - f_0g_+, \quad (173)$$

$$(f \cdot g)_{30} = f_{30}g' + f'g_{30} + f_3g_0 - f_0g_3,$$

$$(f \cdot g)_{0-} = f_{0-}g' + f'g_{0-} + f_0g_- - f_-g_0,$$

$$\begin{aligned} (f \cdot g)_{+-} &= f_{+-}g' + f'g_{+-} + f_+g_- - f_-g_+ + \lambda f_3g_3 \\ &\quad - \lambda f_0g_3, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{+3} &= f_{+3}g' + f'g_{+3} + f_+g_3 - q^{-2}f_3g_+ \\ &\quad - q^{-1}\lambda f_0g_+, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{3-} &= f_{3-}g' + f'g_{3-} + f_3g_- - q^{-2}f_-g_3 \\ &\quad - q^{-1}\lambda f_-g_0, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{+30} &= f_{+30}g' + f'g_{30} + f_+g_{30} - q^{-2}f_3g_{+0} \\ &\quad + f_0g_{+3} - q^{-1}\lambda f_0g_{+0} \\ &\quad + f_{+3}g_0 - f_{+0}g_3 + q^{-2}f_3g_{+}, \quad (174) \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{30-} &= f_{30-}g' + f'g_{30-} + f_3g_{0-} - f_0g_{3-} \\ &\quad + q^{-2}f_-g_{30} + f_{30}g_- - f_{3-}g_0 + q^{-2}f_0-g_3 \\ &\quad + q^{-1}\lambda f_0-g_0, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{+0-} &= f_{+0-}g' + f'g_{+0-} + f_+g_{0-} - f_0g_{+-} \\ &\quad - \lambda f_3g_{30} + f_-g_{+0} - \lambda f_0g_{30} + f_{+0}g_- \\ &\quad - f_{+-}g_0 + f_0-g_+ + \lambda f_3g_3, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{+3-} &= f_{+3-}g' + f'g_{+3-} + q^{-1}\lambda f_+g_{3-} \\ &\quad - q^{-2}f_-g_{+3} - q^{-2}f_3g_{+-} - q^{-1}\lambda f_0g_{+-} \\ &\quad - q^{-1}\lambda f_-g_{+0} - q^{-1}\lambda^2 f_3g_{30} - q^{-1}\lambda^2 f_0g_{30} \\ &\quad + f_{+3}g_- - q^{-2}f_{+-}g_3 + q^{-2}f_{3-}g_+ \\ &\quad - q^{-2}f_{+-}g_3 - q^{-1}\lambda f_{+-}g_0 + q^{-1}\lambda f_0-g_+ \\ &\quad + q^{-1}\lambda(\lambda - q^{-1})f_{30}g_3, \end{aligned}$$

$$\begin{aligned} (f \cdot g)_{+30-} &= f_{+30-}g' + f'g_{+30-} + f_+g_{30-} \\ &\quad - q^{-2}f_3g_{+0-} \quad (175) \\ &\quad + f_0g_{+3-} - q^{-2}f_-g_{+30-} - q^{-1}\lambda f_0g_{+0-} \\ &\quad + f_{+3}g_{0-} - f_{+0}g_{3-} + q^{-2}f_{+-}g_{30} \\ &\quad + q^{-2}f_{30}g_{+-} - q^{-2}f_{3-}g_{+0} + q^{-2}f_0-g_{+3} \\ &\quad + q^{-2}\lambda f_{30}g_{30} + f_{+30}g_- - f_{+3-}g_0 \\ &\quad + q^{-2}f_{+0-}g_3 - q^{-2}f_{30-}g_+ + q^{-1}\lambda f_{+0-}g_0. \end{aligned}$$

Next, we turn to the commutation relations between generators of q -deformed Lorentz algebra (for its definition see [12,44]) and fermionic coordinates. Explicitly, they read

$$T^+\theta^0 = \theta^0T^+, \quad (176)$$

$$T^+\theta^{3/0} = \theta^{3/0}T^+ + q^{-3/2}\lambda_+^{1/2}\theta^+, \quad T^+\theta^+ = q^{-2}\theta^+T^+,$$

$$T^+\theta^- = q^2\theta^-T^+ + q^{-1/2}\lambda_+^{1/2}\theta^3,$$

$$T^-\theta^0 = \theta^0T^-, \quad (177)$$

$$T^-\theta^{3/0} = \theta^{3/0}T^- + q^{3/2}\lambda_+^{1/2}\theta^-, \quad T^-\theta^- = q^2\theta^-T^-,$$

$$\begin{aligned}
T^- \theta^+ &= q^{-2} \theta^+ T^- + q^{1/2} \lambda_+^{1/2} \theta^3, \\
\tau^3 \theta^0 &= \theta^0 \tau^3, \quad \tau^3 \theta^{3/0} = \theta^{3/0} \tau^3, \\
\tau^3 \theta^+ &= q^{-4} \theta^+ \tau^3, \quad \tau^3 \theta^- = q^4 \theta^- \tau^3,
\end{aligned} \tag{178}$$

$$T^2 \theta^{3/0} = q^{-1} \theta^{3/0} T^2, \quad T^2 \theta^+ = q \theta^+ T^2, \tag{179}$$

$$\begin{aligned}
T^2 \theta^- &= q^{-1} \theta^- T^2 + q^{-3/2} \lambda_+^{-1/2} \theta^{3/0} \tau^1, \\
T^2 \theta^3 &= q \theta^3 T^2 - q \lambda_+^{-1} \lambda \theta^{3/0} T^2 + q^{-1/2} \lambda_+^{-1/2} \theta^+ \tau^1, \\
S^1 \theta^{3/0} &= q \theta^{3/0} S^1, \quad S^1 \theta^- = q \theta^- S^1,
\end{aligned} \tag{180}$$

$$\begin{aligned}
S^1 \theta^+ &= q^{-1} \theta^+ S^1 - q^{-1/2} \lambda_+^{-1/2} \theta^{3/0} \sigma^2, \\
S^1 \theta^3 &= q^{-1} \theta^3 S^1 + q^{-1} \lambda_+^{-1} \lambda \theta^{3/0} S^1 - q^{1/2} \lambda_+^{-1/2} \theta^- \sigma^2, \\
\tau^1 \theta^{3/0} &= q \theta^{3/0} \tau^1, \quad \tau^1 \theta^- = q^{-1} \theta^- \tau^1,
\end{aligned} \tag{181}$$

$$\begin{aligned}
\tau^1 \theta^+ &= q \theta^+ \tau^1 - q^{3/2} \lambda_+^{-1/2} \lambda^2 \theta^{3/0} T^2, \\
\tau^1 \theta^3 &= q^{-1} \theta^3 \tau^1 + q^{-1} \lambda_+^{-1} \lambda \theta^{3/0} \tau^1 - q^{1/2} \lambda_+^{-1/2} \lambda^2 \theta^- T^2, \\
\sigma^2 \theta^{3/0} &= q^{-1} \theta^{3/0} \sigma^2, \quad \sigma^2 \theta^+ = q^{-1} \theta^+ \sigma^2,
\end{aligned} \tag{182}$$

$$\begin{aligned}
\sigma^2 \theta^- &= q \theta^- \sigma^2 + q^{1/2} \lambda_+^{-1/2} \lambda^2 \theta^{3/0} S^1, \\
\sigma^2 \theta^3 &= q \theta^3 \sigma^2 - q \lambda_+^{-1} \lambda \theta^{3/0} \sigma^2 + q^{-1/2} \lambda_+^{-1/2} \lambda^2 \theta^+ S^1.
\end{aligned}$$

The generators T^+ , T^- , and τ^3 span a $U_q(su_2)$ -subalgebra of the q -deformed Lorentz algebra. With the above relations at hand we find for its generators the following actions on supernumbers:

$$\tau^3 \triangleright f(\theta^+, \theta^3, \theta^0, \theta^-) = f(q^{-4} \theta^+, \theta^3, \theta^0, q^4 \theta^-) \tag{183}$$

$$T^- \triangleright f(\theta^+, \theta^3, \theta^0, \theta^-) \tag{184}$$

$$\begin{aligned}
&= q^{3/2} \lambda_+^{1/2} f_3 \theta^- + q^{3/2} \lambda_+^{1/2} f_{+3} \theta^+ \theta^- \\
&\quad + q^{1/2} \lambda_+^{1/2} f_{+0} \theta^3 \theta^0 - q^{3/2} \lambda_+^{1/2} f_{30} \theta^0 \theta^- \\
&\quad + q^{1/2} \lambda_+^{1/2} (f_{+-} + \lambda f_{30}) \theta^3 \theta^- \\
&\quad - q^{3/2} \lambda_+^{1/2} f_{3+0} \theta^+ \theta^0 \theta^- + q^{1/2} \lambda_+^{1/2} f_{+0-} \theta^3 \theta^0 \theta^- \\
&\quad + q^{1/2} \lambda \lambda_+^{1/2} f_{+30} \theta^+ \theta^3 \theta^-, \\
T^+ \triangleright f(\theta^+, \theta^3, \theta^0, \theta^-) & \tag{185}
\end{aligned}$$

$$\begin{aligned}
&= q^{-3/2} \lambda_+^{1/2} f_3 \theta^+ - q^{-1/2} \lambda_+^{1/2} f_{0-} \theta^3 \theta^0 \\
&\quad + q^{-3/2} \lambda_+^{1/2} f_{30} \theta^+ \theta^0 + q^{-5/2} \lambda_+^{1/2} f_{+-} \theta^+ \theta^3 \\
&\quad + \lambda_+^{1/2} (q^{1/2} f_{3-} + q^{-1/2} \lambda f_{0-}) \theta^+ \theta^- \\
&\quad - q^{-5/2} \lambda_+^{1/2} f_{+0-} \theta^+ \theta^3 \theta^0 + q^{1/2} \lambda_+^{1/2} f_{30-} \theta^+ \theta^0 \theta^- \\
&\quad - q^{-1/2} \lambda \lambda_+^{1/2} f_{30-} \theta^+ \theta^3 \theta^-.
\end{aligned}$$

Right representations are obtained most easily by either applying the transformations

$$f(\theta^+, \theta^0, \theta^3, \theta^-) \triangleleft T^\pm \tag{186}$$

$$\overleftarrow{\triangleleft} \overrightarrow{\triangleleft} -q^{\mp 3} T^\mp \triangleright f(\theta^+, \theta^3, \theta^0, \theta^-)$$

or the identity

$$f(\theta^+, \theta^0, \theta^3, \theta^-) \triangleleft \tau^3 = f(q^4 \theta^+, \theta^0, \theta^3, q^{-4} \theta^-), \tag{187}$$

where

$$\theta^{\mu_1} \dots \theta^{\mu_n} \overleftarrow{\triangleleft} \overrightarrow{\triangleleft} \theta^{\bar{\mu}_n} \dots \theta^{\bar{\mu}_1}, \tag{188}$$

$$f_{\mu_1 \dots \mu_n} \overleftarrow{\triangleleft} \overrightarrow{\triangleleft} f_{\bar{\mu}_n \dots \bar{\mu}_1},$$

$$f' \overleftarrow{\triangleleft} \overrightarrow{\triangleleft} f',$$

with the conjugated index now defined by

$$\bar{\mu} = \overline{(+, 3, 3/0, 0, -)} = (-, 3, 3/0, 0, +). \tag{189}$$

For the remaining generators we have

$$\sigma^2 \triangleright f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \tag{190}$$

$$\begin{aligned}
&= q^{-1} f_+ \theta^+ + q f_- \theta^- + q f_3 \theta^3 \\
&\quad + (q^{-1} f_{3/0} - q \lambda_+^{-1} \lambda f_3) \theta^{3/0} \\
&\quad + f_{+3} \theta^+ \theta^3 + f_{+-} \theta^+ \theta^- + q^2 f_{3-} \theta^3 \theta^- \\
&\quad + f_{3,3/0} \theta^3 \theta^{3/0} + (f_{3/0,-} - q^2 \lambda \lambda_+^{-1} f_{3-}) \theta^{3/0} \theta^- \\
&\quad + (q^{-2} f_{+,3/0} - \lambda \lambda_+^{-1} f_{+3}) \theta^+ \theta^{3/0} \\
&\quad + q^{-1} f_{+3,3/0} \theta^+ \theta^3 \theta^{3/0} + q f_{+3-} \theta^+ \theta^3 \theta^- \\
&\quad + q f_{3,3/0,-} \theta^3 \theta^{3/0} \theta^- \\
&\quad + (q^{-1} f_{+,3/0,-} - q \lambda \lambda_+^{-1} f_{+3-}) \theta^+ \theta^{3/0} \theta^- \\
&\quad + f_{+3,3/0,-} \theta^+ \theta^3 \theta^{3/0} \theta^-,
\end{aligned}$$

$$\tau^1 \triangleright f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \tag{191}$$

$$\begin{aligned}
&= q^{-1} f_- \theta^- + q f_+ \theta^+ + q^{-1} f_3 \theta^3 \\
&\quad + (q f_{3/0} + q^{-1} \lambda_+^{-1} \lambda f_3) \theta^{3/0} \\
&\quad + f_{+3} \theta^+ \theta^3 + f_{+-} \theta^+ \theta^- + q^{-2} f_{3-} \theta^3 \theta^- \\
&\quad + f_{3,3/0} \theta^3 \theta^{3/0} + q^2 (f_{+3/0} + \lambda \lambda_+^{-1} f_{+3}) \theta^+ \theta^{3/0} \\
&\quad + (f_{3/0,-} + \lambda \lambda_+^{-1} f_{3-}) \theta^{3/0} \theta^- \\
&\quad + q f_{+3,3/0} \theta^+ \theta^3 \theta^{3/0} + q^{-1} f_{+3-} \theta^+ \theta^3 \theta^- \\
&\quad + q^{-1} f_{3,3/0,-} \theta^3 \theta^{3/0} \theta^- \\
&\quad + (q f_{+,3/0,-} + \lambda \lambda_+^{-1} (2(\lambda + q) - q^2) f_{+3-}) \theta^+ \theta^{3/0} \theta^- \\
&\quad + f_{+3,3/0,-} \theta^+ \theta^3 \theta^{3/0} \theta^-,
\end{aligned}$$

$$S^1 \triangleright f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \tag{192}$$

$$\begin{aligned}
&= -q^{-1/2} \lambda_+^{-1/2} f_+ \theta^{3/0} - q^{1/2} \lambda_+^{-1/2} f_3 \theta^- \\
&\quad + q^{1/2} \lambda_+^{-1/2} f_{+3} \theta^3 \theta^{3/0} + q^{-1/2} \lambda_+^{-1/2} (q \lambda - 1) f_{+3} \theta^+ \theta^- \\
&\quad + \lambda_+^{-1/2} (q^{-1/2} f_{3,3/0} - q^{1/2} f_{+-}) \theta^{3/0} \theta^- \\
&\quad + q^{3/2} \lambda_+^{-1/2} f_{+3-} \theta^3 \theta^{3/0} \theta^-
\end{aligned}$$

$$\begin{aligned}
 & +q^{-3/2}\lambda_+^{-1/2}(1-q\lambda)f_{+3,3/0}\theta^+\theta^{3/0}\theta^-, \\
 T^2 \triangleright f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) & \quad (193) \\
 = q^{-3/2}\lambda_+^{-1/2}f_-\theta^{3/0} + q^{-1/2}\lambda_+^{-1/2}f_3\theta^+ & \\
 + q^{-1/2}\lambda_+^{-1/2}f_{3-}\theta^3\theta^{3/0} + q^{-3/2}\lambda_+^{-1/2}f_{3-}\theta^+\theta^- & \\
 + \lambda_+^{-1/2}(q^{1/2}f_{3,3/0} + q^{-1/2}f_{+-})\theta^+\theta^{3/0} & \\
 + q^{1/2}\lambda_+^{-1/2}f_{+3-}\theta^+\theta^3\theta^{3/0} & \\
 + q^{-1/2}\lambda_+^{-1/2}f_{3,3/0,-}\theta^+\theta^3\theta^{3/0}\theta^-. &
 \end{aligned}$$

The easiest way to derive the corresponding right representations is to use the identity

$$S^{-1}(h) \triangleright f = f \triangleleft h \quad (194)$$

together with [44]

$$\begin{aligned}
 S^{-1}(T^2) &= -q^{-2}T^2(\tau^3)^{1/2}, & (195) \\
 S^{-1}(S^1) &= -S^1(\tau^3)^{-1/2}, \\
 S^{-1}(\tau^1) &= \sigma^2, \quad S^{-1}(\sigma^2) = \tau^1.
 \end{aligned}$$

Now, let us consider the differentials, which obey the commutation relations [44]

$$\xi^\mu \xi^\nu = -(\hat{R}_{II})^{\mu\nu}_{\rho\sigma} \xi^\rho \xi^\sigma, \quad \eta^\mu \eta^\nu = q^2(\hat{R}_{II})^{\mu\nu}_{\rho\sigma} \eta^\rho \eta^\sigma \quad (196)$$

and

$$X^\mu \xi^\nu = (\hat{R}_{II})^{\mu\nu}_{\rho\sigma} \xi^\rho X^\sigma, \quad \theta^\mu \eta^\nu = q^2(\hat{R}_{II})^{\mu\nu}_{\rho\sigma} \eta^\rho \theta^\sigma. \quad (197)$$

The Leibniz rules being compatible with the identities in (197) read

$$\begin{aligned}
 \partial_x^\mu X^\nu &= \eta^{\mu\nu} + q^{-2}(\hat{R}_{II}^{-1})^{\mu\nu}_{\rho\sigma} X^\rho \partial_x^\sigma, & (198) \\
 \partial_\theta^\mu \theta^\nu &= \eta^{\mu\nu} - q^{-1}(\hat{R}_{II}^{-1})^{\mu\nu}_{\rho\sigma} \theta^\rho \partial_\theta^\sigma,
 \end{aligned}$$

where $\eta^{\mu\nu}$ stands for the metric of q -deformed Minkowski space. With the substitutions

$$\begin{aligned}
 \partial_a \rightarrow \hat{\partial}_a, \quad a \rightarrow \bar{a}, \quad a \in \{\xi, \eta, X, \theta\}, & \quad (199) \\
 q \rightarrow q^{-1}, \quad \hat{R} \rightarrow \hat{R}^{-1},
 \end{aligned}$$

the formulae in (196)–(198) transform into those of the conjugated calculus.

As in the previous sections, we would like to write down the Leibniz rules for the fermionic derivatives, explicitly. In this way we have

$$\begin{aligned}
 \partial_\theta^{3/0} \theta^{3/0} &= -q^2 \theta^{3/0} \partial_\theta^{3/0}, & (200) \\
 \partial_\theta^{3/0} \theta^+ &= -q^2 \theta^+ \partial_\theta^{3/0} - q\lambda \theta^{3/0} \partial_\theta^+, \\
 \partial_\theta^{3/0} \theta^3 &= 1 - \theta^3 \partial_\theta^{3/0} - \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^{3/0} - \lambda \theta^- \partial_\theta^+, \\
 \partial_\theta^{3/0} \theta^- &= -\theta^- \partial_\theta^{3/0}, \quad \partial_\theta^+ \theta^{3/0} = -\theta^{3/0} \partial_\theta^+, \\
 \partial_\theta^+ \theta^+ &= -q^2 \theta^+ \partial_\theta^+, & (201)
 \end{aligned}$$

$$\begin{aligned}
 \partial_\theta^+ \theta^3 &= -q^2 \theta^3 \partial_\theta^+ + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^+ + q^2 \lambda \lambda_+^{-1} \theta^+ \partial_\theta^{3/0}, \\
 \partial_\theta^+ \theta^- &= -q - \theta^- \partial_\theta^+ + q\lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^{3/0}, \\
 \partial_\theta^- \theta^{3/0} &= -q^2 \theta^{3/0} \partial_\theta^- - q\lambda \theta^- \partial_\theta^{3/0}, & (202)
 \end{aligned}$$

$$\begin{aligned}
 \partial_\theta^- \theta^+ &= -q^{-1} - \theta^+ \partial_\theta^- - \lambda \theta^{3/0} \partial_\theta^0 - \lambda^2 \theta^- \partial_\theta^+ \\
 &\quad - \lambda \theta^3 \partial_\theta^{3/0} - q\lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^{3/0}, \\
 \partial_\theta^- \theta^3 &= -\theta^3 \partial_\theta^- - \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^- - q\lambda \theta^- \partial_\theta^0 \\
 &\quad - \lambda \lambda_+^{-1} (q^2 + 2) \theta^- \partial_\theta^{3/0},
 \end{aligned}$$

$$\begin{aligned}
 \partial_\theta^- \theta^- &= -q^2 \theta^- \partial_\theta^-, \\
 \partial_\theta^0 \theta^{3/0} &= 1 - \theta^{3/0} \partial_\theta^0 - \lambda \theta^- \partial_\theta^+ + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^{3/0}, \\
 \partial_\theta^0 \theta^+ &= -\theta^+ \partial_\theta^0 - q\lambda \theta^3 \partial_\theta^+ + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^+ & (203)
 \end{aligned}$$

$$\begin{aligned}
 &+ q^2 \lambda \lambda_+^{-1} \theta^+ \partial_\theta^{3/0}, \\
 \partial_\theta^0 \theta^3 &= -q^2 \theta^3 \partial_\theta^0 + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^0 + q\lambda \lambda_+^{-1} \theta^+ \partial_\theta^- \\
 &\quad - q\lambda \lambda_+^{-1} \theta^- \partial_\theta^+ + \lambda \lambda_+^{-1} \theta^3 \partial_\theta^{3/0} \\
 &\quad + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^{3/0}, \\
 \partial_\theta^0 \theta^- &= -q^2 \theta^- \partial_\theta^0 - \lambda \lambda_+^{-1} \theta^- \partial_\theta^{3/0} + q^2 \lambda \lambda_+^{-1} \theta^{3/0} \partial_\theta^-.
 \end{aligned}$$

The corresponding expressions for the conjugated calculus follow from the above relations by applying the substitutions

$$\partial_\theta^\mu \rightarrow \hat{\partial}_\theta^\mu, \quad \theta^\mu \rightarrow \bar{\theta}^\mu, \quad q \rightarrow q^{-1}. \quad (204)$$

As usual, the relations in (200)–(203) enable us to compute the action of fermionic derivatives on supernumbers:

$$\begin{aligned}
 \partial_\theta^+ \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) & \quad (205) \\
 = -q f_- - q f_{-+} \theta^+ - q f_{-3} \theta^3 \\
 - q(f_{-,3/0} - \lambda \lambda_+^{-1} f_{-3}) \theta^{3/0} \\
 - q f_{-3+} \theta^3 \theta^+ - q(f_{-,3/0,+} - \lambda \lambda_+^{-1} f_{-3+}) \theta^{3/0} \theta^+ \\
 - q f_{-,3/0,3} \theta^{3/0} \theta^3 - q f_{-,3/0,3+} \theta^{3/0} \theta^3 \theta^+,
 \end{aligned}$$

$$\begin{aligned}
 \partial_\theta^{3/0} \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) & \quad (206) \\
 = f_3 + f_{3+} \theta^+ - q^2 f_{3/0,3} \theta^{3/0} - f_{-3} \theta^- \\
 - q^2 f_{3/0,3+} \theta^{3/0} \theta^+ - f_{-3+} \theta^- \theta^+ + q^2 f_{-,3/0,3} \theta^- \theta^{3/0} \\
 + q^2 f_{-,3/0,3+} \theta^- \theta^{3/0} \theta^+,
 \end{aligned}$$

$$\begin{aligned}
 \partial_\theta^0 \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) & \quad (207) \\
 = f_{3/0} + f_{3/0,3} \theta^3 + (f_{3/0,+} - f_{3+} \lambda \lambda_+^{-1}) \theta^+ \\
 - (q^2 f_{-,3/0} + \lambda \lambda_+^{-1} f_{-3}) \theta^- \\
 - q\lambda \lambda_+^{-1} (f_{-+} - q f_{3/0,3}) \theta^{3/0} \\
 + f_{3/0,3+} \theta^3 \theta^+ + q\lambda f_{3/0,3+} \theta^{3/0} \theta^+ - q^2 f_{-,3/0,3} \theta^- \theta^3 \\
 + q\lambda \lambda_+^{-1} f_{-3+} \theta^{3/0} \theta^3 - q(q f_{-,3/0,+} - \lambda^2 \lambda_+^{-1} f_{-3+}) \theta^- \theta^+
 \end{aligned}$$

$$\begin{aligned}
& -q^2 f_{-,3/0,3+} \theta^- \theta^3 \theta^+ - q^2 \lambda \lambda_+^{-1} f_{-,3/0,3+} \theta^- \theta^{3/0} \theta^+, & + q^{-1/2} \lambda_+^{-1/2} \lambda \Lambda(a) (\tau^1 T^- - q^{-1} S^1) \otimes a^{3/0}, \\
\partial_{\theta^-} \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) & \quad (208) \quad \Delta(a^0) = a^0 \otimes 1 + \Lambda(a) \sigma^2 \otimes a^0 \\
& = -q^{-1} f_+ + q^{-1} f_{3+} \theta^3 + q(f_{+-} - \lambda f_{3/0,3}) \theta^- & - q^{1/2} \lambda_+^{-1/2} \lambda \Lambda(a) T^2 (\tau^3)^{1/2} \otimes a^- \\
& + (q f_{3/0,+} + q^{-1} \lambda \lambda_+^{-1} f_{3+}) \theta^{3/0} & + q^{1/2} \lambda_+^{-1/2} \lambda \Lambda(a) (\tau^3)^{-1/2} (T^- \sigma^2 + q S^1) \otimes a^+ \\
& - q f_{3/0,3+} \theta^{3/0} \theta^3 - q f_{-3+} \theta^- \theta^3 & - \lambda_+^{-1} \Lambda(a) (\lambda^2 T^- T^2 + q(\tau^1 - \sigma^2)) \otimes a^{3/0}, \\
& - q(q^2 f_{-,3/0,+} + \lambda \lambda_+^{-1} f_{-3+}) \theta^- \theta^{3/0} & S(a^{3/0}) = -\Lambda^{-1}(a) \sigma^2 a^{3/0} - q^{-3/2} \lambda_+^{1/2} \lambda \Lambda^{-1}(a) S^1 a^+, \\
& - q \lambda f_{3/0,3+} \theta^- \theta^+ + q^3 f_{-,3/0,3+} \theta^- \theta^{3/0} \theta^3. & S(a^+) = -\Lambda^{-1}(a) \tau^1 (\tau^3)^{1/2} a^+ \quad (215) \\
& & - q^{3/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) T^2 (\tau^3)^{1/2} a^{3/0}, \\
& & S(a^-) = -\Lambda^{-1}(a) \sigma^2 (\tau^3)^{-1/2} a^- \\
& & - q^{-1/2} \lambda_+^{1/2} \lambda \Lambda^{-1}(a) (\tau^3)^{-1/2} S^1 a^0 \\
& & + q^{-2} \lambda^2 \Lambda^{-1}(a) (\tau^3)^{-1/2} S^1 T^- a^+ \\
& & + q^{-5/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) (\tau^3)^{-1/2} \\
& & \quad \times (\sigma^2 T^- - q^3 S^1) a^{3/0}, \\
& & S(a^0) = -\Lambda^{-1}(a) \tau^1 a^0 - q^{5/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) T^2 a^- \\
& & + q^{-3/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) (\tau^1 T^- + q S^1) a^+ \\
& & + \lambda_+^{-1} \Lambda^{-1}(a) (q(\sigma^2 - \tau^1) + \lambda^2 T^2 T^-) a^{3/0}, \\
& & \varepsilon(a^{3/0}) = \varepsilon(a^+) = \varepsilon(a^-) = \varepsilon(a^0) = 0, \quad (216)
\end{aligned}$$

The other types of representations of fermionic derivatives are completely determined by transformation rules of the form

$$\partial_{\theta^0}^\mu \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+) \quad (209)$$

$$\begin{aligned}
& \xrightarrow{q \rightarrow 1/q} \hat{\partial}_{\theta^0}^\mu \triangleright f(\bar{\theta}^+, \bar{\theta}^{3/0}, \bar{\theta}^3, \bar{\theta}^-), \\
q^2 f(\bar{\theta}^-, \bar{\theta}^3, \bar{\theta}^{3/0}, \bar{\theta}^+) \triangleleft \hat{\partial}_{\theta^0}^\mu & \quad (210)
\end{aligned}$$

$$\xrightarrow{q \rightarrow 1/q} q^{-2} f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \triangleleft \partial_{\theta^0}^\mu,$$

and

$$f(\bar{\theta}^-, \bar{\theta}^3, \bar{\theta}^{3/0}, \bar{\theta}^+) \triangleleft \hat{\partial}_{\theta^0}^\mu \quad (211)$$

$$\xrightarrow{q \rightarrow 1/q} q^{-2} \partial_{\theta^0}^\mu \triangleright f(\theta^-, \theta^{3/0}, \theta^3, \theta^+),$$

$$f(\theta^+, \theta^3, \theta^{3/0}, \theta^-) \triangleleft \partial_{\theta^0}^\mu \quad (212)$$

$$\xrightarrow{q \rightarrow 1/q} q^2 \hat{\partial}_{\theta^0}^\mu \triangleright f(\bar{\theta}^+, \bar{\theta}^{3/0}, \bar{\theta}^3, \bar{\theta}^-),$$

where

$$\theta^{i_1} \dots \theta^{i_n} \xrightarrow{q \rightarrow 1/q} \bar{\theta}^{\bar{i}_1} \dots \bar{\theta}^{\bar{i}_n}, \quad (213)$$

$$f_{i_1 \dots i_n} \xrightarrow{q \rightarrow 1/q} \bar{f}_{\bar{i}_1 \dots \bar{i}_n},$$

$$f' \xrightarrow{q \rightarrow 1/q} \bar{f}',$$

$$q \xrightarrow{q \rightarrow 1/q} q^{-1}.$$

Finally, we come to the Hopf structure for the quantum spaces of the q -deformed Lorentz algebra. In general, we have

$$\Delta(a^{3/0}) = a^{3/0} \otimes 1 + \Lambda(a) \tau^1 \otimes a^{3/0} \quad (214)$$

$$- q^{1/2} \lambda_+^{1/2} \lambda \Lambda(a) (\tau^3)^{-1/2} S^1 \otimes a^+,$$

$$\Delta(a^+) = a^+ \otimes 1 + \Lambda^{1/2} (\tau^3)^{-1/2} \sigma^2 \otimes a^+$$

$$- q^{3/2} \lambda_+^{-1/2} \lambda \Lambda(a) T^2 \otimes a^{3/0},$$

$$\Delta(a^-) = a^- \otimes 1 + \Lambda(a) (\tau^3)^{1/2} \tau^1 \otimes a^-$$

$$- q^{-1/2} \lambda_+^{1/2} \lambda \Lambda(a) S^1 \otimes a^0$$

$$- \lambda^2 \Lambda(a) (\tau^3)^{-1/2} T^- S^1 \otimes a^+$$

$$\begin{aligned}
& \bar{\Delta}(a^{3/0}) = a^{3/0} \otimes 1 + \Lambda^{-1}(a) (\tau^3)^{1/2} \sigma^2 \otimes a^{3/0} \quad (217) \\
& - q^{3/2} \lambda_+^{1/2} \lambda \Lambda^{-1}(a) T^2 \otimes a^-, \\
& \bar{\Delta}(a^-) = a^- \otimes 1 + \Lambda^{-1}(a) \tau^1 \otimes a^- \\
& - q^{1/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) (\tau^3)^{-1/2} S^1 \otimes a^{3/0}, \\
& \bar{\Delta}(a^+) = a^+ \otimes 1 + \Lambda^{-1}(a) \sigma^2 \otimes a^+ \\
& - q^{1/2} \lambda_+^{1/2} \lambda \Lambda^{-1}(a) T^2 (\tau^3)^{1/2} \otimes a^0 \\
& - q^{1/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) (\tau^3)^{-1/2} (T^+ \sigma^2 + q \tau^3 T^2) \\
& \quad \otimes a^{3/0} \\
& + q^2 \lambda^2 \Lambda^{-1}(a) T^2 T^+ \otimes a^-, \\
& \bar{\Delta}(a^0) = a^0 \otimes 1 + \Lambda^{-1}(a) (\tau^3)^{-1/2} \tau^1 \otimes a^0 \\
& - q^{-1/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) S^1 \otimes a^+ \\
& - q^{1/2} \lambda_+^{-1/2} \lambda \Lambda^{-1}(a) (q T^+ \tau^1 - T^2) \otimes a^- \\
& + \lambda_+^{-1} \Lambda^{-1}(a) (\tau^3)^{-1/2} \\
& \quad \times (\lambda^2 T^+ S^1 + q^{-1} (\tau^3 \tau^1 - \sigma^2)) \otimes a^{3/0}, \\
& \bar{S}(a^{3/0}) = -\Lambda(a) \tau^1 (\tau^3)^{1/2} a^{3/0} \\
& - q^{3/2} \lambda_+^{1/2} \lambda \Lambda(a) T^2 (\tau^3)^{1/2} a^-, \quad (218)
\end{aligned}$$

$$\begin{aligned}
 \bar{S}(a^-) &= -\Lambda(a)\sigma^2 a^- - q^{-3/2}\lambda_+^{-1/2}\lambda\Lambda(a)S^1 a^{3/0}, \\
 \bar{S}(a^+) &= -\Lambda(a)\tau^1 a^+ - q^{5/2}\lambda_+^{1/2}\lambda\Lambda(a)T^2 a^0 \\
 &\quad - q^{3/2}\lambda_+^{-1/2}\lambda\Lambda(a)(q\tau^1 T^+ + T^2)a^{3/0} \\
 &\quad - q^4\lambda^2\Lambda(a)T^2 T^+ a^-, \\
 \bar{S}(a^0) &= -\Lambda(a)(\tau^3)^{-1/2}\sigma^2 a^0 \\
 &\quad - q^{-1/2}\lambda_+^{-1/2}\lambda\Lambda(a)(\tau^3)^{-1/2}S^1 a^+ \\
 &\quad - q^{3/2}\lambda_+^{-1/2}\lambda\Lambda(a)(\tau^3)^{-1/2}(\sigma^2 T^+ - q\tau^3 T^2)a^- \\
 &\quad - \lambda_+^{-1}\Lambda(a)(\tau^3)^{-1/2} \\
 &\quad \quad \times (\lambda^2 T^+ S^1 + q(\sigma^2 - \tau^3 \tau^1)) a^{3/0}, \\
 \bar{\varepsilon}(a^{3/0}) &= \bar{\varepsilon}(a^+) = \bar{\varepsilon}(a^-) = \bar{\varepsilon}(a^0) = 0,
 \end{aligned} \tag{219}$$

with

$$a \in \{\partial_x, \partial_\theta, X, \theta, \xi, \eta\}. \tag{220}$$

The scaling operators have to take the form

$$\Lambda(\partial_x^i) = \Lambda^{1/2}, \quad \Lambda(X^i) = \Lambda^{-1/2}, \quad \Lambda(\eta^i) = \Lambda^{-1}, \tag{221}$$

or

$$\Lambda(\partial_\theta^i) = \tilde{\Lambda}^{-1}, \quad \Lambda(\theta^i) = \tilde{\Lambda}, \quad \Lambda(\xi^i) = \tilde{\Lambda}, \tag{222}$$

if the operators Λ and $\tilde{\Lambda}$ are subject to the relations

$$\begin{aligned}
 \Lambda X^\mu &= q^{-2} X^\mu \Lambda, & \tilde{\Lambda} X^\mu &= q^{-1} X^\mu \tilde{\Lambda}, \\
 \Lambda \partial_x^\mu &= q^2 \partial_x^\mu \Lambda, & \tilde{\Lambda} \partial_x^\mu &= q \partial_x^\mu \tilde{\Lambda}, \\
 \Lambda \xi^\mu &= q^{-2} \xi^\mu \Lambda, & \tilde{\Lambda} \xi^\mu &= -q^{-1} \xi^\mu \tilde{\Lambda}, \\
 \Lambda \eta^\mu &= q^{-4} \eta^\mu \Lambda, & \tilde{\Lambda} \eta^\mu &= q^2 \eta^\mu \tilde{\Lambda}, \\
 \Lambda \theta^\mu &= q^{-2} \theta^\mu \Lambda, & \tilde{\Lambda} \theta^\mu &= -q^{-1} \theta^\mu \tilde{\Lambda}, \\
 \Lambda \partial_\theta^\mu &= q^2 \partial_\theta^\mu \Lambda, & \tilde{\Lambda} \partial_\theta^\mu &= -q \partial_\theta^\mu \tilde{\Lambda}.
 \end{aligned} \tag{223}$$

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Appendix A: Representations of supernumbers

In this article we deal with supernumbers of different normal orderings. Thus, it can be useful to have formulae at hand that allow to switch between the different orderings. For this purpose we wish to list the following identities.

(1) Two-dimensional Euclidean space:

$$\begin{aligned}
 f' + f_1\theta^1 + f_2\theta^2 + f_{12}\theta^1\theta^2 \\
 = \tilde{f}' + \tilde{f}_1\theta^1 + \tilde{f}_2\theta^2 + \tilde{f}_{21}\theta^2\theta^1,
 \end{aligned} \tag{224}$$

where

$$\tilde{f}' = f', \quad \tilde{f}_1 = f_1, \quad \tilde{f}_2 = f_2, \quad \tilde{f}_{21} = -q^{-1}f_{12}. \tag{225}$$

(2) Three-dimensional Euclidean space:

$$\begin{aligned}
 f' + f_+\theta^+ + f_3\theta^3 + f_-\theta^- + f_{+3}\theta^+\theta^3 + f_{+-}\theta^+\theta^- \\
 + f_{3-}\theta^3\theta^- + f_{+3-}\theta^+\theta^3\theta^- \\
 = \tilde{f}' + \tilde{f}_+\theta^+ + \tilde{f}_3\theta^3 + \tilde{f}_-\theta^- + \tilde{f}_{3+}\theta^3\theta^+ + \tilde{f}_{-+}\theta^-\theta^+ \\
 + \tilde{f}_{-3}\theta^-\theta^3 + \tilde{f}_{-3+}\theta^-\theta^3\theta^+,
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{f}' &= f', \quad \tilde{f}_A = f_A, \quad A \in \{+, 3, -\} \\
 \tilde{f}_{-+} &= -f_{+-}, \quad \tilde{f}_{3+} = -q^2 f_{+3}, \quad \tilde{f}_{-3} = -q^2 f_{3-}, \\
 \tilde{f}_{-3+} &= -q^4 f_{+3-}.
 \end{aligned} \tag{226}$$

(3) Four-dimensional Euclidean space:

$$\begin{aligned}
 f' + \sum_{i=1}^4 f_i\theta^i + \sum_{1 \leq i_1 < i_2 \leq 4} f_{i_1 i_2}\theta^{i_1}\theta^{i_2} \\
 + \sum_{1 \leq i_1 < i_2 < i_3 \leq 4} f_{i_1 i_2 i_3}\theta^{i_1}\theta^{i_2}\theta^{i_3} + f_{1234}\theta^1\theta^2\theta^3\theta^4 \\
 = \tilde{f}' + \sum_{i=1}^4 \tilde{f}_i\theta^i + \sum_{1 \leq i_2 < i_1 \leq 4} \tilde{f}_{i_1 i_2}\theta^{i_1}\theta^{i_2} \\
 + \sum_{1 \leq i_3 < i_2 < i_1 \leq 4} \tilde{f}_{i_1 i_2 i_3}\theta^{i_1}\theta^{i_2}\theta^{i_3} + \tilde{f}_{4321}\theta^4\theta^3\theta^2\theta^1,
 \end{aligned} \tag{227}$$

where

$$\begin{aligned}
 \tilde{f}' &= f', \quad \tilde{f}_i = f_i, \quad i = 1, \dots, 4, \\
 \tilde{f}_{21} &= -q^{-1}f_{12}, \quad \tilde{f}_{31} = -q^{-1}f_{13}, \\
 \tilde{f}_{41} &= -f_{14} - \lambda f_{23}, \quad \tilde{f}_{32} = -f_{23}, \quad \tilde{f}_{42} = -q^{-1}f_{24}, \\
 \tilde{f}_{43} &= -q^{-1}f_{34}, \quad \tilde{f}_{321} = -q^{-2}f_{123}, \\
 \tilde{f}_{421} &= -q^{-2}f_{124}, \quad \tilde{f}_{431} = -q^{-2}f_{134}, \\
 \tilde{f}_{432} &= -q^{-2}f_{234}, \quad \tilde{f}_{4321} = q^{-4}f_{1234}.
 \end{aligned} \tag{228}$$

(4) Minkowski space:

$$\begin{aligned}
 f' + f_+\theta^+ + f_{3/0}\theta^{3/0} + f_3\theta^3 + f_-\theta^- + f_{+,3/0}\theta^+\theta^{3/0} \\
 + f_{+3}\theta^+\theta^3 + f_{+-}\theta^+\theta^- + f_{3/0,3}\theta^{3/0}\theta^3 + f_{3-}\theta^3\theta^- \\
 + f_{3/0,3}\theta^{3/0}\theta^3 + f_{3/0,-}\theta^{3/0}\theta^- + f_{+,3/0,3}\theta^+\theta^{3/0}\theta^3 \\
 + f_{+3-}\theta^+\theta^3\theta^- + f_{+,3/0,-}\theta^+\theta^{3/0}\theta^- \\
 + f_{3/0,3-}\theta^{3/0}\theta^3\theta^- + f_{+,3/0,3-}\theta^+\theta^{3/0}\theta^3\theta^- \\
 = \tilde{f}' + \tilde{f}_+\theta^+ + \tilde{f}_{3/0}\theta^{3/0} + \tilde{f}_3\theta^3 + \tilde{f}_-\theta^- \\
 + \tilde{f}_{3/0,+}\theta^{3/0}\theta^+ + \tilde{f}_{3+}\theta^3\theta^+ + \tilde{f}_{-+}\theta^-\theta^+ \\
 + \tilde{f}_{3,3/0}\theta^3\theta^{3/0} + \tilde{f}_{-3}\theta^-\theta^3 + \tilde{f}_{-,3/0}\theta^-\theta^{3/0} \\
 + \tilde{f}_{3,3/0,+}\theta^3\theta^{3/0}\theta^+ + \tilde{f}_{-3+}\theta^-\theta^3\theta^+ + \tilde{f}_{-,3/0,+}\theta^-\theta^{3/0}\theta^+
 \end{aligned}$$

$$+\tilde{f}_{-3,3/0}\theta^{-}\theta^3\theta^{3/0} + \tilde{f}_{-3,3/0,+}\theta^{-}\theta^3\theta^{3/0}\theta^+,$$

where

$$\begin{aligned} \tilde{f}' &= f', & \tilde{f}_\mu &= f_\mu, & \mu &\in \{+, 3/0, 3, -\}, & (229) \\ \tilde{f}_{+3} &= -q^{-2}f_{3+}, & \tilde{f}_{+,3/0} &= -f_{3/0,+}, \\ \tilde{f}_{+-} &= -f_{-+} + \lambda f_{3/0,3}, & \tilde{f}_{3,3/0} &= -f_{3/0,3}, \\ \tilde{f}_{3-} &= -q^{-2}f_{-3}, & \tilde{f}_{3/0,-} &= -f_{-,3/0} \\ \tilde{f}_{+3,3/0} &= -q^2f_{3/0,3+}, & \tilde{f}_{+3-} &= -q^{-4}f_{-3+}, \\ \tilde{f}_{+,3/0,-} &= -f_{-,3/0,+}, & \tilde{f}_{3,3/0,-} &= -q^{-2}f_{-,3/0,3}, \\ \tilde{f}_{+3,3/0,-} &= q^{-4}f_{-,3/0,3+}. \end{aligned}$$

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